

Lecture 10 - End of universal algebra

- First part: two general theorems
- Second part: Birkhoff's Theorem

- Last week:
 - $\mathcal{R}\text{-Alg}$ has colimits
 - Inclusion $(\mathcal{R}, E)\text{-Alg} \hookrightarrow \mathcal{R}\text{-Alg}$ has a left adjoint.
 - Forgetful functor $U: (\mathcal{R}, E)\text{-Alg} \rightarrow \text{Set}$ has a left adjoint.

Theorem $(\mathcal{R}, E)\text{-Alg}$ has all colimits.

Proof Consider $(\mathcal{R}, E)\text{-Alg} \xleftarrow[\underset{i}{\perp}]{R} \mathcal{R}\text{-Alg}$.

- Consider $D: J \rightarrow (\mathcal{R}, E)\text{-Alg}$. We claim it has colimit $R \text{col}(iD)$ where $\text{col}(iD)$ is colimit of $J \xrightarrow{D} (\mathcal{R}, E)\text{-Alg}$ $\xrightarrow{i} \mathcal{R}\text{-Alg}$
- There is a bijection (nat. in X) between

- ① cones $(D_j \rightarrow X)_{j \in J}$ (as $(\mathcal{R}, E)\text{-Alg}$ is full subcategory)
- ② cones $(iD_j \rightarrow iX)_{j \in J}$
- ③ morphisms $\text{col}(iD) \rightarrow iX$ (by univ. prop of $\text{col}(iD)$)
- ④ morphisms $R \text{col}(iD) \rightarrow X$ (using adjunction $R \dashv i$)

Note: starting at ④ with $R \text{col}(iD) \xrightarrow{1} R \text{col}(iD)$ gives in ① cone $(D_j \rightarrow R \text{col}(iD))_{j \in J}$ through which each other cone factors uniquely (by nat. in X); this shows it

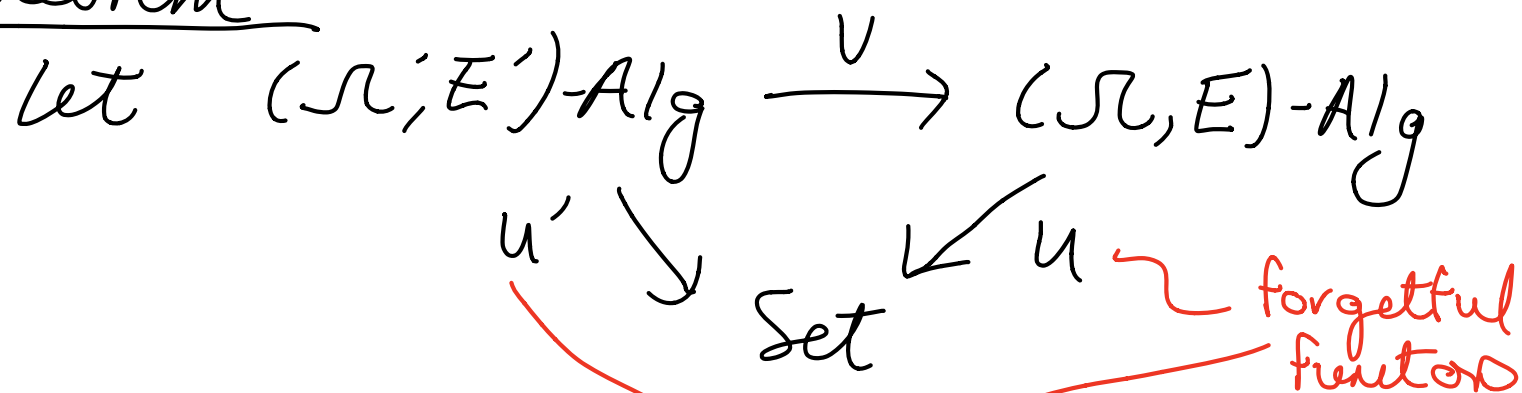
has univ. prop. as in first part of course.

□

In summary $(\Omega, E)\text{-Alg}$ has all limits & colimits.

• A very general result is the following one - I will only outline the proof, which is quite categorical.

Theorem



commute. Then U has a left adjoint.

Remark: This says "any forgetful functor between algebraic categories has a left adjoint".

Eg: $(\Omega, E)\text{-Alg} \xrightarrow{\text{Forgetful}} \Omega\text{-Alg}$
 $(\Omega, E)\text{-Alg} \xrightarrow{\text{Forgetful}} \text{Set}$
• $U: \text{Ring} \longrightarrow \text{Ab}, \text{Ab} \longrightarrow \text{Grp},$

$S\text{-Mod} \rightarrow R\text{-Mod}$ for any ring hom
 $R \rightarrow S \dots$

Proof (Sketch)

Let's say $A \in (\Omega, E)\text{-alg}$ has a reflection
along V if $\exists R A \in (\Omega', E')\text{-Alg}$ &
 $\eta: A \rightarrow V R A$ st. given $A \xrightarrow{F} V B$
 $\exists ! \bar{F} R A \rightarrow B$ such that

$$\begin{array}{ccc}
 A & \xrightarrow{\eta} & V R A \\
 & \searrow F & \downarrow V \bar{F} \\
 & & V B
 \end{array}$$

• Then V has a left adjoint \Leftrightarrow
each A has a reflection along V .

$$\begin{array}{ccc}
 (\Omega', E')\text{-Alg} & \xrightarrow{V} & (\Omega, E)\text{-Alg} \\
 u' \downarrow & & \downarrow u \\
 & \text{Set} &
 \end{array}$$

• Now we have $F' \dashv u'$ & $F \dashv u$ & the
free algebra $F X$ has reflection
 $F' X$ along V .

• Moreover, full subcat
 $V\text{-Ref} \hookrightarrow (\Omega, E)\text{-Alg}$ containing

the objects admitting V -reflection
 is closed under colimits
 in $(\Omega, E)\text{-Alg}$ (exercise)
 (Hint: same idea as left adj. preserve colims.)

• So it suffices to show that each (Ω, E) -algebra is a colimit of free algebras: in fact, we show it is a coequaliser of free algebras.

• let $A \in (\Omega, E)\text{-Alg}$. Then
 $\exists! FUA \xrightarrow{\epsilon_A} A \in (\Omega, E)\text{-Alg}$
 st $\begin{array}{ccc} \eta_{UA} & \xrightarrow{UFUA} & U\epsilon_A \\ & \searrow & \downarrow \\ & UA & \xrightarrow{\quad} UA \end{array}$

Therefore $\epsilon_A: FUA \rightarrow A$ is surjective
 as $\epsilon_A(\eta_A(x)) = x$.

• So by First iso. theorem,
 it is coequaliser of ϵ_A its kernel
 $\text{ker}(\epsilon_A) \xrightleftharpoons[c]{d} FUA \xrightarrow{\epsilon_A} A$

• Now consider $Fu\ker(\epsilon_A) \xrightarrow{\epsilon_K} \ker(\epsilon_A)$
 which is also surjective (& so epi)
 & it follows that the coequaliser

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \underbrace{Fu\ker(\epsilon_A)} & \xrightarrow[\quad c\epsilon_K]{\quad d\epsilon_K} & \underbrace{FUA} \\ \text{Free alg} & & \text{Free alg} \end{array}$$

is the same as coequaliser
 of d, c - namely, A .

Therefore A is a coequaliser
 of free algebras as required. \square

Part 2 - Birkhoff's Theorem

Question) Which full subcategories $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$ are of the form $(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ for a set E of equations?

Answer) Last week: Full subcats $(\Omega, E)\text{-Alg}$ are closed under (H)omomorphic images, (P)roducts and (S)ubalgebras -

- in fact, we will see that $\mathcal{C} = (\Omega, E)\text{-Alg} \Leftrightarrow$ it has these closure properties.

This is Birkhoff's theorem: relates syntax (equations) in terms of semantics (sets of algebras).

- In course so far, an equation means a pair $s, t \in \text{Tr } X$ for some X . In fact, it is enough to consider equations only in a fixed countable set of variables

$$\omega = \{x_1, x_2, \dots, x_n, \dots\}$$

Lemma Let $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$. TFAE

- ① $\mathcal{C} = (\Omega, E)\text{-Alg}$ for E set of eqs in variables ω
- ② $\mathcal{C} = (\Omega, E)\text{-Alg}$ for E set of equations,
- ③ $\mathcal{C} = (\Omega, E)\text{-Alg}$ for E class of equations.

Proof Suffices to show $(3 \Rightarrow 1)$ & for this to show

* given $s, t \in \text{Tr } X \exists \bar{s}, \bar{t} \in \text{Tr } \omega$ st

$$A \models s = t \iff A \models \bar{s} = \bar{t}$$

Indeed, then $(\Omega, E)\text{-Alg} = (\Omega, \bar{E})\text{-Alg}$
 where $\bar{E} = \{ (\bar{s}, \bar{t}) \in (T_{\Omega} \omega)^2 : (s, t) \in E \}$
 & there is only a set of such.

• Firstly, let $f: X \rightarrow Y$ be injective &
 write $f^*: F_{\Omega} X \rightarrow F_{\Omega} Y$ for value
 under left adjoint F_{Ω} .

• We firstly prove that

\otimes given $s, t \in T_{\Omega} X$, $A \models s = t \iff A \models f^*(s) = f^*(t)$.

- If $A \models s = t$, let $F_{\Omega} Y \xrightarrow{p} A$. Must show $p f^*(s) = p f^*(t)$
 But $p f^*$ a hom, so $p f^*(s) = p f^*(t)$.

Conversely, $A \models f^*s = f^*t$ means given
 $F_{\Omega} Y \xrightarrow{p} A$ we have $p f^*s = p f^*t$.

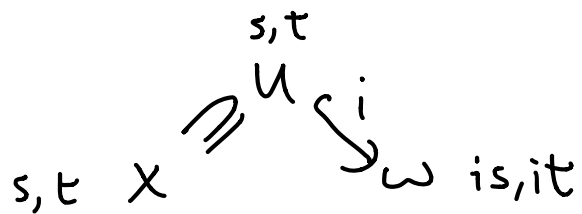
- If A is empty, \exists no homomorphisms
 $F_{\Omega} X \rightarrow A$ so $A \models s = t$ trivially.

- Otherwise $X \xrightarrow{f} Y$
 it contains $v \searrow \text{" } A \swarrow \exists w$ by setting
 $\omega f x = v x$ &
 $\omega y = a$ else.

Then $F_{\Omega} X \xrightarrow{f} F_{\Omega} Y$
 $\bar{v} \searrow \text{" } A \swarrow \bar{w}$ so

given $\bar{v}: F_{\Omega} X \rightarrow A$ we have
 $\bar{v} s = \bar{w} f^* s = \bar{w} f^* t = \bar{v} t$ as
 required.

- - Now let $s, t \in \text{Tr} X$. We will define a finite subset $U \subseteq X$ st. $s, t \in \text{Tr} U \subseteq \text{Tr} X$.
- Given this, take any inj. function $i: U \rightarrow \omega$ we have



$$\begin{array}{l}
 \text{by } (*) \\
 \text{by } (*)
 \end{array}
 \begin{array}{l}
 \Leftrightarrow \\
 \Leftarrow
 \end{array}
 \begin{array}{l}
 A \models s = t \text{ in vars } X \\
 A \models s = t \text{ in vars } U \\
 A \models is = it \text{ in vars } \omega.
 \end{array}$$

- What is U ? Well given $s \in \text{Tr} X$ we can inductively define its (finite) set of variables:

$$\text{var}(x) = \{x\}$$

$$\text{var}(f(t_1, \dots, t_n)) = \text{var}t_1 \cup \dots \cup \text{var}t_n \subseteq X.$$

- Then $s \in \text{Tr}(\text{vars})$.

- So $s, t \in \text{Tr}(\text{vars} \cup \text{var}t)$ so take $U = \text{vars} \cup \text{var}t$.

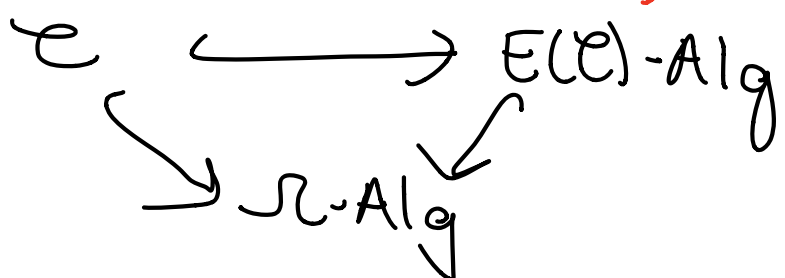
□

- As a result, we will allow E to be a class of equations.

- Given a full subcategory $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$, we define $E(\mathcal{C})$ the class of equations sat. by objects of \mathcal{C} ; namely,

$$E(\mathcal{C}) = \{ (s, t) : \text{if } A \in \mathcal{C} \text{ then } A \models s = t \}$$

- Then we have $\mathcal{C} \hookrightarrow E(\mathcal{C})\text{-Alg}$ (also. For $(\Omega, E(\mathcal{C}))\text{-Alg}$)



since if $A \in \mathcal{C}$, then it trivially satisfies any equation satisfied by all objects of \mathcal{C} .

- Write $H\mathcal{C} =$ closure of \mathcal{C} in $\Omega\text{-Alg}$ under homomorphic images.
- $S\mathcal{C} =$ closure - - - - - under subalgebras.
- $P\mathcal{C} =$ closure - - - - - products.

Eg: $HSP(\mathcal{C})$ closure under all 3 of Ω

these, again $HSP(\mathcal{C}) \hookrightarrow \mathcal{R}\text{-Alg}$.

Theorem For any full subcat

$\mathcal{C} \hookrightarrow \mathcal{R}\text{-Alg}$, we have

$E(\mathcal{C})\text{-Alg} = HSP(\mathcal{C})$, the closure of \mathcal{C} in $\mathcal{R}\text{-Alg}$ under prods, h. ins, subalgebras.

Proof • As above, $\mathcal{C} \subseteq E(\mathcal{C})\text{-Alg}$. From last week, $E(\mathcal{C})\text{-Alg}$ closed in $\mathcal{R}\text{-Alg}$ under H, S, P.

Therefore $HSP(\mathcal{C}) \subseteq HSP(E(\mathcal{C})\text{-Alg}) = E(\mathcal{C})\text{-Alg}$.

• Conversely, must show

$$E(\mathcal{C})\text{-Alg} \subseteq HSP(\mathcal{C})$$

• Let $A \in E(\mathcal{C})\text{-Alg}$. From last week have adjunction $E(\mathcal{C})\text{-Alg} \begin{matrix} \xleftarrow{F} \\ \xrightarrow{U} \end{matrix} \text{Set}$

& as in first part of lecture,

the counit map $FUA \xrightarrow{\epsilon_A} A$ (ind. by $UA \xrightarrow{1} UA$ & u.p. of FUA) is surjective as sat $\epsilon_A(\eta_{UA}(x)) = x$ for $x \in A$ where η unit of adj.

• Therefore A is homomorphic image of a free $E(\mathcal{C})$ -algebra,

so it suffices to show each free algebra FX belongs to $SP(\mathcal{C})$; since then

$$A \in HSP(\mathcal{C}).$$

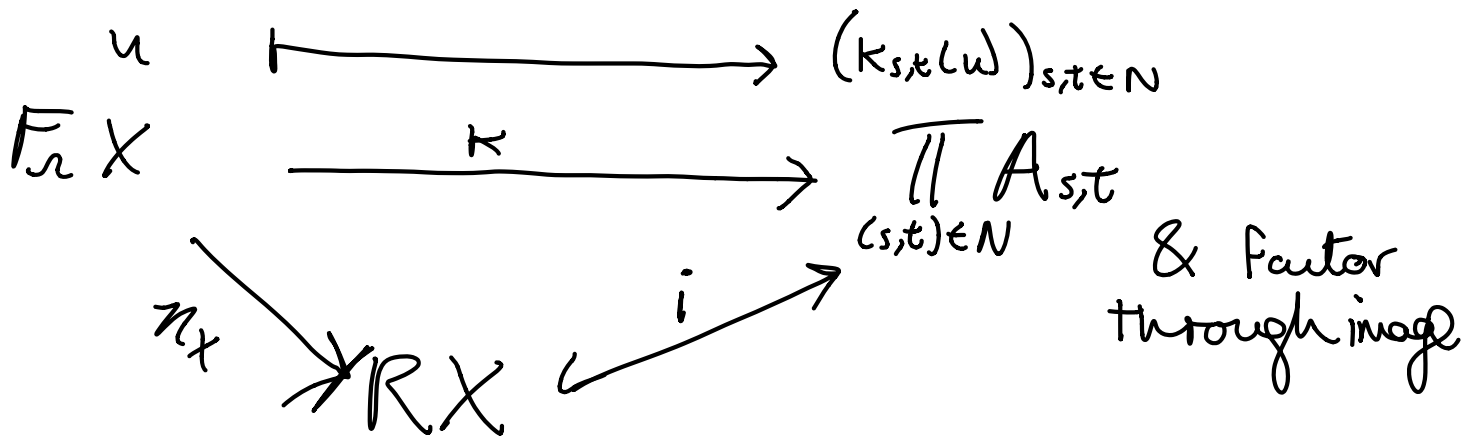
- To this end, let $FrX = \text{free } \Omega\text{-algebra on } X$.

Let $N = \{ (s, t) \in FrX : (s, t) \notin E(\mathcal{C}) \}$.

Then given $(s, t) \in N$,

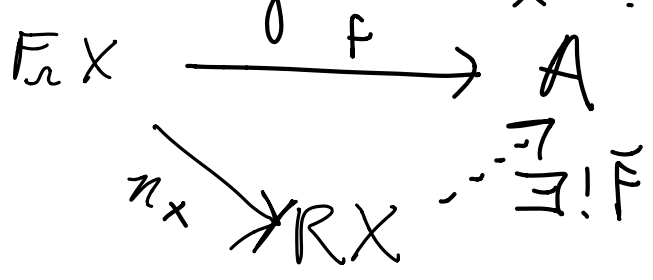
$\exists FrX \xrightarrow{K_{s,t}} A_{s,t} \in \mathcal{C}$ such that $K_{s,t}(s) \neq K_{s,t}(t)$.

- Consider the induced maps to product



Then $RX \in HSP(\mathcal{C}) \subseteq E(\mathcal{C})\text{-Alg}$. It remains to prove that RX is free $E(\mathcal{C})$ -algebra on X : for this, it is enough to show

that if A is an $E(\mathcal{C})$ -alg, then each $FrX \xrightarrow{f} A \in \Omega\text{-Alg}$ Factor uniquely through n_x :



since then

RX	\longrightarrow	$A \in E(\mathcal{C})\text{-Alg}$	w_{ij}
FrX	\longrightarrow	$A \in \Omega\text{-Alg}$	b_{ij}
X	\longrightarrow	$U_{\Omega} A = UA \in \text{Set}$	

Let $(s,t) \in \text{Fr}(X)^2$.

Now if $(s,t) \in N$, then $ks \neq kt$ since they are unequal in (s,t) -component $k_{s,t}$.

On other hand, if $(s,t) \in E(C)$, then $ks = kt$ as $\prod A_{s,t} \in \text{HSP}(C) \subseteq E(C)\text{-Alg}$.
Therefore $ks = kt \Leftrightarrow (s,t) \in E(C)$.

• Since $k = i \circ \pi_x$ & i inj therefore $\pi_x s = \pi_x t \Leftrightarrow (s,t) \in E(C)$,
 where $\text{Fr} X \xrightarrow{\pi_x} RX$.

Therefore $\text{Ker}(\pi_x) = \{(s,t) \in \text{Fr} X : (s,t) \in E(C)\}$.

Since π_x is surjective, by the first isomorphism theorem, it is the quotient of its kernel:

therefore given $\text{Fr} X \xrightarrow{f} B$ s.t. $f(s) = f(t)$ for each $(s,t) \in E(C) \exists!$

