Lecture 10 - End of universal algebra

- First part: two general theorems
- Second part: Birkhoff's theorem.

- Last week: - Ω-Alg has colimits
  - Inclusion $(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ has a left adjoint.
  - Forgetful functor $U : (\Omega, E)\text{-Alg} \to \text{Set}$ has a left adjoint.

$\boxed{\text{Theorem}}$ (Ω, E)-Alg has all colimits.

$\boxed{\text{Proof}}$ Consider
$$\begin{tikzcd} (\Omega,E)\text{-Alg} \arrow[r, hook, shift right, "i"'] \arrow[r, phantom, "\perp"] & \Omega\text{-Alg} \arrow[l, shift right, "R"'] \end{tikzcd}.$$

- Consider $D : J \to (\Omega, E)\text{-Alg}$. We claim it has colimit $R\,\mathrm{col}(iD)$ where $\mathrm{col}(iD)$ is colimit of $J \xrightarrow{D} (\Omega,E)\text{-Alg} \xrightarrow{i} \Omega\text{-Alg}$
- There is a bijection (nat. in X) between

① cocones $(D_j \to X)_{j \in J}$ (as (Ω,E)-Alg is full subcategory)

② cocones $(iD_j \to iX)_{j \in J}$ (by univ. prop of $\mathrm{col}(iD)$)

③ morphisms $\mathrm{col}(iD) \to iX$ (using adjunction $R \dashv i$.)

④ morphisms $R\,\mathrm{col}(iD) \to X$

Note: starting at ④ with $R\,\mathrm{col}(iD) \xrightarrow{1} R\,\mathrm{col}(iD)$ gives in ① cocone $(D_j \to R\,\mathrm{col}(iD))_{j \in J}$ through which each other cocone factors uniquely (by nat. in X); this shows it has univ. prop. as in first part of course.

$\square$

- In summary (Ω, E)-Alg has all limits & colimits.

- A very general result is the following one – I will only outline the proof, which is quite categorical.

Theorem

Let

$$\begin{tikzcd}
(\Omega', E')\text{-Alg} \arrow[rr, "V"] \arrow[dr, "U'"'] & & (\Omega, E)\text{-Alg} \arrow[dl, "U"] \\
& \mathrm{Set} &
\end{tikzcd}$$

forgetful functors

commute. Then V has a left adjoint.

Remark: This says "any forgetful functor between algebraic categories has a left adjoint".

Eg:
- $(\Omega, E)\text{-Alg} \longrightarrow \Omega\text{-Alg}$
- $(\Omega, E)\text{-Alg} \xrightarrow{\text{forgetful}} \mathrm{Set}$
- $U : \mathrm{Rng} \longrightarrow \mathrm{Ab}$, $\mathrm{Ab} \longrightarrow \mathrm{Grp}$,

$S\text{-Mod} \to R\text{-Mod}$ for any ring hom $R \to S$ ...

$\boxed{\text{Proof}}$ (Sketch)

Let's say $A \in (\Omega, E)$-alg has a reflection along V if $\exists RA \in (\Omega', E')$-Alg & $\eta : A \to VRA$ s.t. given $A \xrightarrow{f} VB$ $\exists! \bar{f}\, RA \to B$ such that

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, bend right, "f"'] & VRA \arrow[d, "V\bar{f}"] \\ & VB \end{tikzcd}$$

- Then V has a left adjoint $\Leftrightarrow$ each A has a reflection along V.

- $$\begin{tikzcd} (\Omega', E')\text{-Alg} \arrow[rr, "V"] \arrow[dr, "U'"'] & & (\Omega, E)\text{-Alg} \arrow[dl, "U"] \\ & \text{Set} & \end{tikzcd}$$

- Now we have $F' \dashv U'$ & $F \dashv U$ & the free algebra FX has reflection $F'X$ along V.

- Moreover, full subcat $V\text{-Ref} \hookrightarrow (\Omega, E)\text{-Alg}$ containing the objects admitting V-reflection is closed under colimits in (Ω, E)-Alg (exercise)

(Hint: same idea as left adj. preserve colims.)

- So it suffices to show that each (Ω, E)-algebra is a colimit of free algebras: in fact, we show it is a coequaliser of free algebras.

- Let $A \in (\Omega, E)$-Alg. Then $\exists!\ FUA \xrightarrow{\varepsilon_A} A \in (\Omega, E)$-Alg st

$$\begin{tikzcd} & UFUA \arrow[dr, dashed, "U\varepsilon_A"] \arrow[d, phantom, "="] & \\ UA \arrow[ur, "\eta_{UA}"] \arrow[rr] & & UA \end{tikzcd}$$

Therefore $\varepsilon_A : FUA \longrightarrow A$ is surjective as $\varepsilon_A(\eta_A(x)) = x$.

- So by First iso. theorem, it is coequaliser of its kernel

$$\ker(\varepsilon_A) \underset{c}{\overset{d}{\rightrightarrows}} FUA \xrightarrow{\varepsilon_A} A.$$

• Now consider $FU\ker(\varepsilon_A) \xrightarrow{\varepsilon_K} \ker(\varepsilon_A)$ which is also surjective (& so epi) & it follows that the coequaliser of

$$\overbrace{FU\ker(\varepsilon_A)}^{\text{free alg}} \underset{c\varepsilon_K}{\overset{d\varepsilon_K}{\rightrightarrows}} \overbrace{FUA}^{\text{free alg}}$$

is the same as coequaliser of d, c — namely, A.

Therefore A is a coequaliser of free algebras as required. $\square$

Part 2 - Birkhoff's Theorem

Question) Which full subcategories $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$ are of the form $(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ for a set E of equations?

Answer) Last week: full subcats $(\Omega, E)\text{-Alg}$ are closed under (H)omomorphic images, (P)roducts and (S)ubalgebras -

- in fact, we will see that $\mathcal{C} = (\Omega, E)\text{-Alg} \Leftarrow$ it has these closure properties.

This is Birkhoff's theorem: relates syntax (equations) in terms of semantics (cats of algebras).

- In course so far, an equation means a pair $s, t \in T_\Omega X$ for some X. In fact, it is enough to consider equations only in a fixed countable set of variables $\omega = \{x_1, x_2, ---, x_n, -- \}$.

Lemma Let $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$. TFAE

1. $\mathcal{C} = (\Omega, E)\text{-Alg}$ for E set of eqs in variables ω
2. $\mathcal{C} = (\Omega, E)\text{-Alg}$ for E set of equations,
3. $\mathcal{C} = (\Omega, E)\text{-Alg}$ for E class of equations.

Proof Suffices to show $(3 \Rightarrow 1)$ & for this to show

(*) given $s, t \in T_\Omega X$ $\exists \bar{s}, \bar{t} \in T_\Omega \omega$ st

$A \models s = t \iff A \models \bar{s} = \bar{t}$.

Indeed, then (Ω, E)-Alg $= (\Omega, \bar{E})$-Alg where $\bar{E} = \{ (\bar{s}, \bar{t}) \in (T_\Omega \omega)^2 : (s,t) \in E \}$ & there is only a set of such.

- Firstly, let $f : X \to Y$ be injective & write $f^* : F_\Omega f : F_\Omega X \to F_\Omega Y$ for value under left adjoint F_Ω.
- We firstly prove that

(*) given $s, t \in T_\Omega X$, $A \models s = t \iff A \models f^*(s) = f^*(t)$.

- If $A \models s = t$, let $F_\Omega Y \xrightarrow{p} A$. Must show $pf^*(s) = pf^*(t)$.

But pf^* a hom, so $pf^*(s) = pf^*(t)$.

Conversely, $A \models f^*s = f^*t$ means given

$F_\Omega Y \xrightarrow{p} A$ we have $pf^*s = pf^*t$.

- If A is empty, $\exists$ no homomorphisms $F_\Omega X \to A$ so $A \models s = t$ trivially.
- Otherwise it contains $a \in A$ so for $v : X \to A$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "v"'] & Y \arrow[d, phantom, "="] \arrow[dl, dashed, "\exists w"] \\ & A \end{tikzcd}$$

by setting $wfx = vx$ & $wy = a$ else.

Then

$$\begin{tikzcd} F_\Omega X \arrow[r, "f"] \arrow[dr, "\bar{v}"'] & F_\Omega Y \arrow[d, phantom, "="] \arrow[dl, "\bar{w}"] \\ & A \end{tikzcd}$$

so

given $\bar{v} : F_\Omega X \to A$ we have

$\bar{v}s = \bar{w}f^*s = \bar{w}f^*t = \bar{v}t$ as required.

- Now let $s,t \in \mathrm{Tr}X$. We will define a finite subset $U \subseteq X$ s.t. $s,t \in \mathrm{Tr}U \subseteq \mathrm{Tr}X$.
- Given this, take any inj. function $i: U \to \omega$ we have

$$\begin{tikzcd} & \overset{s,t}{U} \arrow[dl, phantom, "\supseteq"] \arrow[dr, hook, "i"] & \\ s,t \; X & & \omega \; is, it \end{tikzcd}$$

$$\begin{aligned} & A \models s=t \text{ in vars } X \\ \text{by } (*) \iff \; & A \models s=t \text{ in vars } U \\ \text{by } (*) \iff \; & A \models is = it \text{ in vars } \omega. \end{aligned}$$

- What is U? Well given $s \in \mathrm{Tr}X$ we can inductively define its (finite) set of variables:

$$\mathrm{var}(x) = \{x\}$$
$$\mathrm{var}(f(t_1,\dots,t_n)) = \mathrm{var}\,t_1 \cup \dots \cup \mathrm{var}\,t_n \subseteq X.$$

- Then $s \in \mathrm{Tr}(\mathrm{var}\,s)$.
- So $s,t \in \mathrm{Tr}(\mathrm{var}\,s \cup \mathrm{var}\,t)$ so take

$$U = \mathrm{var}\,s \cup \mathrm{var}\,t.$$

$\square$

- As a result, we will allow E to be a class of equations.

- Given a full subcategory $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$, we define $E(\mathcal{C})$ the class of equations sat. by objects of $\mathcal{C}$; namely,

$$E(\mathcal{C}) = \{ (s,t) : \text{if } A \in \mathcal{C} \text{ then } A \models s = t \}.$$

- Then we have

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr, hook] & E(\mathcal{C})\text{-Alg} \arrow[d, hook] \\ & \Omega\text{-Alg} \end{tikzcd}$$

~ (abb. for $(\Omega, E(\mathcal{C}))\text{-Alg}$)

since if $A \in \mathcal{C}$, then it trivially satisfies any equation satisfied by all objects of $\mathcal{C}$.

- Write $H\mathcal{C}$ = closure of $\mathcal{C}$ in Ω-Alg under homomorphic images.
- $S\mathcal{C}$ = closure . . . . under subalgebras.
- $P\mathcal{C}$ = closure . . . . products,

Eg: $HSP(\mathcal{C})$ closure under all 3 of these, again $HSP(\mathcal{C}) \hookrightarrow \Omega\text{-Alg}$.

$\boxed{\text{Theorem}}$ For any full subcat $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$, we have $E(\mathcal{C})\text{-Alg} = HSP(\mathcal{C})$, the closure of $\mathcal{C}$ in $\Omega\text{-Alg}$ under prods, h.ims, subalgebras.

$\boxed{\text{Proof}}$ • As above, $\mathcal{C} \subseteq E(\mathcal{C})\text{-Alg}$. From last week, $E(\mathcal{C})\text{-Alg}$ closed in $\Omega\text{-Alg}$ under H, S, P.

Therefore $HSP(\mathcal{C}) \subseteq HSP(E(\mathcal{C})\text{-Alg}) = E(\mathcal{C})\text{-Alg}$.

• Conversely, must show $E(\mathcal{C})\text{-Alg} \subseteq HSP(\mathcal{C})$

- Let $A \in E(\mathcal{C})\text{-Alg}$. From last week have adjunction

$$E(\mathcal{C})\text{-Alg} \underset{U}{\overset{F}{\rightleftarrows}} \text{Set} \quad (\perp)$$

& as in first part of lecture, the counit map $FUA \xrightarrow{\varepsilon_A} A$ (ind. by $UA \xrightarrow{1} UA$ & u.p. of FUA) is surjective as sat $\varepsilon_A(\eta_{UA}(x)) = x$ for $x \in A$ where η unit of adj.

- Therefore A is homomorphic image of a free $E(\mathcal{C})$-algebra, so it suffices to show each free algebra FX belongs to $SP(\mathcal{C})$; since then $A \in HSP(\mathcal{C})$.

- To this end, let $F_\Omega X$ = free Ω-algebra on X.

Let $N = \{(s,t) \in F_\Omega X : (s,t) \notin E(\mathcal{C})\}$.

Then given $(s,t) \in N$,

$\exists\ F_\Omega X \xrightarrow{k_{s,t}} A_{s,t} \in \mathcal{C}$ such that $k_{s,t}(s) \neq k_{s,t}(t)$.

- Consider the induced map to product

$$u \longmapsto (k_{s,t}(u))_{s,t \in N}$$

$$\begin{tikzcd} F_\Omega X \arrow[rr, "k"] \arrow[dr, two heads, "\eta_X"'] & & \prod\limits_{(s,t)\in N} A_{s,t} \\ & RX \arrow[ur, hook, "i"'] & \end{tikzcd}$$

& factor through image

Then $RX \in HSP(\mathcal{C}) \subseteq E(\mathcal{C})$-Alg. It remains to prove that RX is free $E(\mathcal{C})$-algebra on X: for this, it is enough to show that if A is an $E(\mathcal{C})$-alg, then each $F_\Omega X \xrightarrow{f} A \in \Omega$-Alg factors uniquely through η_X:

$$\begin{tikzcd} F_\Omega X \arrow[rr, "f"] \arrow[dr, two heads, "\eta_X"'] & & A \\ & RX \arrow[ur, dashed, "\exists ! \bar{f}"'] & \end{tikzcd}$$

since then

$$\frac{\dfrac{RX \longrightarrow A \in E(\mathcal{C})\text{-Alg}}{F_\Omega X \longrightarrow A \in \Omega\text{-Alg}}\ \text{bij}}{X \longrightarrow U_\Omega A = UA \in \mathbf{Set}}\ \text{bij}$$

Let $(s,t) \in F_R(X)^2$.

Now if $(s,t) \in N$, then $ks \neq kt$ since they are unequal in (s,t)-component $k_{s,t}$.

On other hand, if $(s,t) \in E(\mathcal{C})$, then $ks = kt$ as $\prod A_{s,t} \in HSP(\mathcal{C}) \subseteq E(\mathcal{C})$-Alg.

Therefore $ks = kt \iff (s,t) \in E(\mathcal{C})$.

- Since $k = i \circ n_X$ & i inj therefore $n_X s = n_X t \iff (s,t) \in E(\mathcal{C})$, where $F_R X \xrightarrow{n_X} RX$.

Therefore $\ker(n_X) = \{(s,t) \in F_R X : (s,t) \in E(\mathcal{C})\}$.

Since n_X is surjective, by the First isomorphism theorem, it is the quotient of its kernel:

therefore given $F_R X \xrightarrow{f} B$ s.t. $f(s) = f(t)$ for each $(s,t) \in E(\mathcal{C})$ $\exists!$ $\bar{f} : RX \longrightarrow B$ making

$$\begin{tikzcd} F_R X \arrow[r, "f"] \arrow[dr, two heads, "n_X"'] & B \\ & RX \arrow[u, "\bar{f}"'] \end{tikzcd}$$

commute.

In particular, if B is an $E(\mathcal{C})$-algebra then f has this prop. so we obtain the unique factorisation. $\square$

Birkhoff's Theorem

$\mathcal{C} \hookrightarrow \Omega\text{-Alg}$ is of the form $(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg} \iff$ $\mathcal{C}$ closed under hom. images, products & subalgebras.

Proof

By last week (Ω,E)-Alg has these closure props. Conversely, $\mathcal{C} = HSP(\mathcal{C}) = (\Omega, E(\mathcal{C}))\text{-Alg}$ using previous theorem. $\square$