Exercises 10

(1) In the course we have looked at inclusions $A \xhookrightarrow{i} B$ of full subcategories, eg.

$$(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}.$$

- Show that if $i : A \hookrightarrow B$ has left adjoint R, then the counit $RiX \xrightarrow{\varepsilon_X} X$ is an iso.

(Recall: the counit is unique map such that

$$\begin{tikzcd} & iRiX \arrow[dr, "i\varepsilon_X"] & \\ iX \arrow[ur, "\eta_{iX}"] \arrow[rr, "1"] & & iX \end{tikzcd}$$

commutes.)

- Such an R is called a reflection, and A a reflective subcat.

2. Let $f: R \to S$ be a ring homomorph.

It induces $f^*: S\text{-Mod} \to R\text{-Mod}$

$$(A, \cdot) \longmapsto (A, *) \text{ where}$$

$$r * a := fr \cdot a$$

Using results from the course, show f^* has a left adjoint.

- Think of some other left adjoints to "algebraic functors".

3

Check that if $A \xrightarrow{f} B$ is epi then the coequaliser of $B \underset{h}{\overset{g}{\rightrightarrows}} C$ & $A \underset{hf}{\overset{gf}{\rightrightarrows}} C$ coincide.

Using this,
show that each (Ω, E)-alg is a coequaliser of free algebras.

4) Completing proof that algebraic functors have left adjoints.

- For $U : A \longrightarrow B$ recall that $x \in B$ has a U-reflection if $\exists x' \in A$ & $x \xrightarrow{\eta} Ux'$ such that: given $x \xrightarrow{f} Uy$ $\exists!$ $x' \xrightarrow{\bar{f}} y$ such that

$$\begin{tikzcd} & Ux' \arrow[d, "U\bar{f}"] \\ x \arrow[ur, "\eta"] \arrow[r, "f"'] & Uy \end{tikzcd}$$

commutes.

- Suppose A, B have J-colimits. Then show that the objects in B admitting a U-reflection are closed under J-colimits.