Lecture 11 - Module Theory

Def) Let R be a ring. A left R-module $(M, +, 0)$ is an abelian group with a function $\cdot \, R \times M \longrightarrow M$ satisfying:

$$\left.\begin{array}{l} \cdot \; r.(a+b) = r.a + r.b \\ \cdot \; (r+s).a = r.a + s.a \end{array}\right\} \text{bilinearity}$$

$$\left.\begin{array}{l} \cdot \; (r.s).a = r.(s.a) \\ \cdot \; 1.a = a \end{array}\right\} \text{action equations.}$$

Remark: Also right R-modules: but these are left R^{op}-modules where R^{op} is R w/ same abelian group str. but multip. $(x,y) \longmapsto y.x$.
So suffices to develop theory of left R-modules for most purposes.

- A homomorphism $f: M \longrightarrow N$ of R-modules is a homomorphism of abelian groups (i.e. $f(a+b) = fa + fb$) such that $f(r.a) = r.f(a)$.
- There is a category Mod_R of left R-modules, which has forgetful functor $U : \mathrm{Mod}_R \to \mathrm{Set}$.

Examples

① When $R = K$ is a field, a K-module is a K-vector space.

② When $K = \mathbb{Z}$, a $\mathbb{Z}$-module is precisely an abelian group.

Indeed, if M is an ab. group, we are forced to define $\cdot : \mathbb{Z} \times M \longrightarrow M$ as follows: since each $-\cdot m : \mathbb{Z} \longrightarrow M$ is homom. of abelian groups & since $\mathbb{Z}$ is free ab. gp. on 1, we must def.

$$n \cdot m = (\underbrace{1 + \dots + 1}_{n \text{ times}}) \cdot m = \underbrace{m + \dots + m}_{n \text{ times}}$$

& sim. for negative n.

Exercise: check remaining details.

---

③ Let G be a group & R a ring.

We can form the group ring $R[G] :=$

$\{ \lambda_1 g_1 + \dots + \lambda_n g_n : \lambda_1, \dots, \lambda_n \in R, g_1, \dots, g_k \in G \}$

set of formal linear combinations of elements of G.

- This is clearly an abelian group with multiplication

$$\left(\sum_{i=1}^{k} \lambda_i g_i\right)\left(\sum_{j=1}^{l} \mu_j g_j\right) = \sum_{i=1}^{k} \sum_{j=1}^{l} (\lambda_i \mu_j)(g_i g_j)$$

mult in R mult. in G

& unit $e \in G$, the group unit.

- $R[G]$-modules are often called group representations:
  they amount to R-modules M with
  $\cdot : G \times M \longrightarrow M$ satisfying
  $e.m = m$ &
  $(g.h).m = g.(h.m)$

Exercise: check this!

This is of most interest when $R = K$ is a field & the vector space M is fin. dim., so $M = K^n$;
then a $K[G]$-module amounts to a group homomorphism

$$G \longrightarrow \underbrace{GL(n,K)}_{\text{group of inv. } n\times n\text{-matrices w/ values in } K}$$

This is subject of group representation theory.

Remark: $\mathrm{Mod}_R = (\Omega, E)\text{-Alg}$
where $\Omega = \{ r \cdot - : r \in R, +, 0 \}$

unary operation for each $r \in R$ — binary — nullary

with equations E as in the def^n of an R-module.

- So study of modules is part of universal algebra.

So, from previous lectures, it follows that Mod_R has limits, colimits, forgetful functors
$$\mathrm{Mod}_R \longrightarrow \mathrm{Ab}, \mathrm{Set} \text{ have}$$
left adjoints.

Kernels & quotients

- As for abelian groups, kernels & quotients for modules are very simple.

Def^n) • Given $f : M \longrightarrow N \in \mathrm{Mod}_R$, its kernel $\ker(f) \hookrightarrow M$ is the submodule
$\ker f = \{x \in M : fx = 0\}$.

- Given a submodule $N \subseteq M$, the quotient $M/N = \{m+N : m \in M\}$ is quotient abelian group (set of cosets) with $r.(m+N) = rm+N$. We then have $M \xrightarrow{p_N} M/N$ module homomorphism.

Exercise:
Express $\ker f$ & M/N as equaliser & coequaliser respectively.

• The first isomorphism theorem in this setting says:

Theorem

① If $f : M \longrightarrow N \in \mathrm{Mod}_R$, then $M/\ker f \cong \mathrm{im} f$. In particular, if f is surjective, then $M/\ker f \cong N$.

(2) If $N \subseteq M$ a submodule, then
$N = \ker(M \xrightarrow{p_M} M/N)$.

- This is all routine. More interesting about Mod_R is behaviour of products & coproducts.

Products & coproducts

Proposition
$0 = \{0\}$ is both the terminal & initial object in Mod_R.

Proof
Clearly terminal. 0 is initial as must have $f: 0 \to M$ by $f(0) = 0$.

Remark: The zero homomorphism $0: M \to N$ is the composite

$$\begin{array}{ccccc} M & \xrightarrow{\exists!} & 0 & \xrightarrow{\exists!} & N \\ x & \longmapsto & 0 & \longmapsto & 0 \end{array}$$

, which exists as 0 is term. & initial (a zero object)

- Given a set $(M_i)_{i\in I}$ of R-modules, then (as for any algebraic cat) the product $\prod_{i\in I} M_i \xrightarrow{p_i} M_i$ consists of sequences $(a_i)_{i\in I}$ where $a_i \in M_i$, with component-wise R-module structure: ie. $r.(a_i)_{i\in I} = (r.a_i)_{i\in I}$.

Def^n - The direct sum $\bigoplus_{i\in I} M_i \hookrightarrow \prod_{i\in I} M_i$ is the submodule containing those $(a_i)_{i\in I}$ for which $a_i \neq 0$ only for finitely many $i \in I$

Remark: In particular, if I is finite, then $\bigoplus_{i\in I} M_i = \prod_{i\in I} M_i$.

- Observe that there are R-module homom's:

$$\begin{aligned} M_i &\xrightarrow{q_i} \bigoplus_{i\in I} M_i \\ a &\longmapsto q_i(a) \text{ sequence with } (q_i(a))_i = a \ \&\ 0 \text{ otherwise.} \end{aligned}$$

Theorem

The maps $M_i \xrightarrow{q_i} \bigoplus_{i\in I} M_i$ exhibit the direct sum as the coproduct.

Proof

Consider $(f_i : M_i \longrightarrow N)_{i\in I}$ a set of homomorphisms in Mod_R.

- Must show $\exists!\, f \in Mod_R$ making

$$\begin{tikzcd} & \bigoplus_{i\in I} M_i \arrow[dr, "f"] & \\ M_i \arrow[ur, "q_i"] \arrow[rr, "f_i"'] & & N \end{tikzcd}$$

- Given $a = (a_i)_{i\in I} \in \bigoplus M_i$.

Then $a = q_{i_1}(a_{i_1}) + \ldots\ldots + q_{i_n}(a_{i_n})$ where $i_1, \ldots, i_n \in I$ are the elements at which a is non-zero.

Therefore, for f to be a homorphism sat. $f \circ q_i = f_i$, we must define
$f(a) = f_{i_1}(a_{i_1}) + \ldots + f_{i_n}(a_{i_n})$

- It is straightforward to see that f is a R-module hom. using that N is abelian group.

Indeed, given $a, b \in \bigoplus M_i$, let $i_1, \ldots, i_n, j_1, \ldots, j_m$ be elts of I at which a, b respectively non-zero.

- Then $f(a+b) =$
$f_{i_1}(a_{i_1} + b_{i_1}) + \ldots + f(a_{j_m} + b_{j_m}) =$
$f_{i_1} a_{i_1} + f_{i_1} b_{i_1} + \ldots + f_{j_m} a_{j_m} + f_{j_m} b_{j_m}$
$= f_{i_1} a_{i_1} + \ldots + f_{i_n} a_{i_n} +$
$f_{j_1} b_{j_1} + \ldots + f_{j_m} b_{j_m}$ as N is abelian
$= f(a) + f(b)$.

- Easy to see $f(r.a) = r.fa$.

$\square$

- In particular, $A \oplus B = A \times B$ is both product & coproduct.
- Similarly, R^n is both product & coproduct in Mod-R?

Remark) This explains correspondence between homomorphisms

$$R^m \xrightarrow{f} R^n \quad \&$$

$m \times n$-matrices of elements of R.

- Indeed $f : R^m \longrightarrow R^n \in \mathrm{Mod}_R$

$R \xrightarrow{f_i} R^n$ for $i = \{1, \dots, m\}$ as R^m is m-fold coproduct

---

$R \xrightarrow{f_{ij}} R$ for $i = \{1, \dots, m\}$ & $j = \{1, \dots, n\}$ as R^n is n-fold prod.

Elements $\{A_{ij}\}$ for $i = \{1, \dots, m\}$ & $j = \{1, \dots, n\}$ as R is free module on 1 element (i.e. $f_{ij} : R \longrightarrow R$ is uniquely spec. by value A_{ij} at 1)

- In other words, a matrix

- Under this correspondence, composition of homomorphisms corresponds to matrix multiplication.