Lecture 13 – Projective & injective modules

Def^n) An R-module M is projective if: given $f : A \to B$ surj, & $g : M \to B$ then $\exists g' : M \to A$ such that

$$\begin{tikzcd} & M \arrow[dl, "g'"'] \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}.$$

Proposition

Each free module is projective.

Proof

Let FX be free & consider

$$\begin{tikzcd} & FX \arrow[d, "g"] \\ A \arrow[r, "f \text{ surj}"'] & B \end{tikzcd}.$$

Then we have

$$\begin{tikzcd} & X \arrow[d, "\bar{g}"] \\ UA \arrow[r, "Uf"'] & UB \end{tikzcd}$$

by $F \dashv U$.

Since Uf is surjective function, $\exists s : UB \to UA$ s.t. $Uf . s = 1$. Therefore

$$\begin{tikzcd} & X \arrow[dl, "t = s . \bar{g}"'] \arrow[dr, "\bar{g}"] \arrow[d, phantom, "="] & \\ UA \arrow[rr, "Uf"'] & & UB \end{tikzcd}$$

& then by $F \dashv U$ again,

$$\begin{tikzcd} & FX \arrow[dl, "\bar{t}"'] \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}.$$

$\square$

We will prove that that projective modules $\equiv$ retracts of free modules.

Def^n) In a category $\mathcal{C}$, we say A is a retract of B if $\exists\, A \underset{f}{\overset{g}{\rightleftarrows}} B$ w/ $gf = 1_A$.

Theorem

Projective modules $\equiv$ retracts of frees.

Proof

We know free $\Rightarrow$ projective. So for one direction, sufficient to show that retracts of projective modules are proj.

- Consider $A \underset{f}{\overset{g}{\rightleftarrows}} B$ w/ $gf = 1$ & B proj.
- Consider $M \overset{p}{\twoheadrightarrow} N$ surj &

$$\begin{tikzcd} & A \arrow[d, "k"] \\ M \arrow[r, two heads, "p"'] & N \end{tikzcd}$$

Then as B proj. $\exists \bar{k}$ as in

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "\bar{k}"'] \arrow[dr, phantom, "="] & A \arrow[d, "k"] \\ M \arrow[r, two heads, "p"'] & N \end{tikzcd}$$

& then take

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "1"] \arrow[ddr, bend left=60, "k"] & \\ B \arrow[r, "g"] \arrow[d, "\bar{k}"'] \arrow[dr, phantom, "="] & A \arrow[d, "k"] \\ M \arrow[r, two heads, "p"'] & N \end{tikzcd}$$

k as required.

- Conversely, let M be projective. Then by the univ. prop. of FUM, $\exists!\ FUM \xrightarrow{\varepsilon_M} M$ s.t.

$$\begin{tikzcd} & UFUM \arrow[dr, "U\varepsilon_M"] & \\ UM \arrow[ur, "\eta_{UM}"] \arrow[rr, "1"'] & = & UM \end{tikzcd}$$

The same diagram shows ε_M is surj.

Therefore as M is projective,

$$\exists k \quad \begin{tikzcd} & M \arrow[dl, dashed] \arrow[d, "1"] \\ FUM \arrow[r, two heads, "\varepsilon_M"'] & M \end{tikzcd} \quad \text{s.t. } \varepsilon_M . k = 1$$

so M is a retract of FUM, as required.

<u>Example</u>

Consider ring $\mathbb{Z}_6$. Then $\mathbb{Z}_6$ is free $\mathbb{Z}_6$-module &

$$\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \text{ as modules.}$$

Since direct summands are retracts

$$\text{eg.} \quad \begin{tikzcd} a & (a,b) \arrow[l, mapsto] \\ \mathbb{Z}_2 \arrow[r, shift right] & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \arrow[l, shift right] \\ a \arrow[r, mapsto] & (a,0) \end{tikzcd}$$

we see, by prev. theorem, that $\mathbb{Z}_2, \mathbb{Z}_3$ are projective (though not free.)

Injective modules

- Dual to projective modules.

Def^n) A module M is injective if for each mono $i : A \hookrightarrow B$ & $A \xrightarrow{f} M$ $\exists B \xrightarrow{\bar{f}} M$ st

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "f"'] & B \arrow[d, "\bar{f}"] \\ & M \end{tikzcd}$$

$=$.

- To understand inj modules in practise, the following is very useful.

Theorem (Baer criterion)

M is injective $\iff$ it has the extension property wrt submodules $I \hookrightarrow R$ (ie. ideals of R)

Proof

- It suff. to show that M has ext. property wrt submodule inclusion $A \hookrightarrow B$.
- Consider

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"'] & B \\ M & \end{tikzcd}$$

- Consider pairs (B_i, f_i) where $A \subseteq B_i \subseteq B$ submod. homom.
& $f_i : B_i \to M$ st $f_i | A = f$.
- Write $(B_i, f_i) \leq (B_j, f_j)$ if $B_i \subseteq B_j$
& $f_j | B_i = f_i$

$$\left(\text{i.e.}\quad \begin{tikzcd} A \arrow[r, hook] \arrow[drr, bend right, "f"'] & B_i \arrow[r, hook] \arrow[dr, "f_i"] & B_j \arrow[r, hook] \arrow[d, "F_j"] & B \\ & & M & \end{tikzcd} \right)$$

- The set of such pairs is a poset
- Will use Zorn's lemma to show it has a maximal element:

so consider a chain $(B_i, f_i)_{i \in I}$ of submodules of B & maps f_i, & form $\bigcup_{i \in I} B_i = \{ x \in B : \exists i \in I, x \in B_i \} \subseteq B$.

Then $\bigcup_{i \in I} B_i$ is submodule, & can define homomorphism $f^* : \bigcup_{i \in I} B_i \longrightarrow M$ by

$$x \longmapsto f_i(x) \text{ if } x \in B_i.$$

This is a well def. hom. extending f. Thus the chain has an upper bound $\Rightarrow$ by Zorn the poset has a max^{e} element (N, g) as pictured:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] & N \arrow[r, hook] \arrow[d, "g"] & B \\ & M & \end{tikzcd}$$

- Suppose that $N \neq B$. Will show this leads to a contradiction, compl. the proof.
- Indeed let $b \in B - N$ & consider

$$I = \{ r \in R : rb \in N \} \subseteq R.$$

In fact
$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "-\cdot b"'] \arrow[dr, phantom, "\lrcorner"] & R \arrow[d, "-\cdot b"] & r \arrow[d, mapsto] \\ N \arrow[r, hook] & B & rb \end{tikzcd}$$
is a pullback.

- Then have
$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] & R \arrow[d] \\ N \arrow[r, hook] & N+Rb \subseteq B \end{tikzcd}$$
the submodule $N+Rb = \{n+rb : n \in N, r \in R\} \subseteq B$.

Real idea: this is a pushout!

- If we can show that $g: N \longrightarrow M$ extends to $N+Rb$, this will also extend f & so contradict maximality of N, completing the proof.

- Consider
$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "-\cdot b"'] & R \arrow[d, "-\cdot b"] \arrow[ddl, bend left, "h"] \\ N \arrow[r, hook] \arrow[d, "g"'] & N+Rb \\ M & \end{tikzcd}$$
$\exists h$ by assumption

- Define $N+Rb \xrightarrow{\ell} M$
$\ell(n+rb) = gn + hr$
which clearly extends g.

- Must show ℓ is a module map: firstly, well defined:

- suppose $n+rb = n'+r'b \in N+Rb$.

- Then $(r-r')b = n'-n$, so $r-r' \in I$.

Then $l(n+rb) - l(n'+r'b) =$

$gn + hr - gn' - hr' =$

$g(n-n') + h(r-r')$ & $r-r' \in I$

$= g(n-n') + g((r-r')b) = 0$

as $(r-r')b = n-n'$.

- More easily, l is a homomorph.

$\square$

<u>Corollary</u>

In Ab, A is injective $\iff$ it is <u>divisible</u>: that is,

$\forall a \in A$ & $n \neq 0 \in \mathbb{N}$

$\exists b \in A$ st $nb = a$.

<u>Proof</u>

- Here $R = \mathbb{Z}$ ring of integers.
- Ideals of $\mathbb{Z}$ are principal

$$n\mathbb{Z} \hookrightarrow \mathbb{Z}.$$

Consider

$$\begin{tikzcd} n\mathbb{Z} \arrow[r, "f"] \arrow[d, hook] & A \\ \mathbb{Z} & \end{tikzcd}$$

- A map f is spec. by $a \in A$ st. $f(n) = a$.
- To give an extension $g : \mathbb{Z} \to A$ of f along $n\mathbb{Z} \hookrightarrow \mathbb{Z}$ is to give $g(1) = b \in A$ st

$$a = g(n) = \underbrace{g(1) + \ldots + g(1)}_{n \text{ times}}$$
$$= n.b$$

Then by Baer criterion, A is inj $\Leftrightarrow$ all such extensions exist - that is, just when A is divisible. $\square$

Projective & injective resolutions

- In homological alg, it is important that each module M has a proj. & inj resolution.
- For this, key points are:

① Given M we can find proj. M' & surjection $M' \twoheadrightarrow M$.

② - - - - - - - . inj. M' & injection $M \rightarrowtail M'$.

Prop ① is easy (take $FUM \overset{\varepsilon_M}{\twoheadrightarrow} M$).

② I will sketch now.

- Special case of so-called small object argument:

We are given some set J of morphisms $j : A \to B$ in a cat $\mathcal{C}$, and we want to find for each X, a map

$$X \xrightarrow{p} X^{\#} \text{ with}$$

$X^{\#}$ J-injective:

given $j : A \longrightarrow B \in J$

&

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X^{\#} \\ B \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Example: For Mod_R, we take submodules $I \hookrightarrow R$ & obtain injectivity.

How to construct $X^{\#}$?

As colimit of a chain

$$\begin{tikzcd} X = X_0 \arrow[r] \arrow[drrrr, bend right, "p = p_0"'] & X_1 \arrow[r] \arrow[drrr, bend right] & \cdots \arrow[r] & X_n \arrow[r, "p^n_{n+1}"] \arrow[dr, "p_n"'] & X_{n+1} \arrow[r] \arrow[d, "p_{n+1}"] & \cdots \\ & & & & X^{\#} & \end{tikzcd}$$

where $X = X_0$ & where X_{n+1} contains extensions for lifting problems in X_n: more precisely,

$\forall$ $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j \in J"'] & X_n \\ B & \end{tikzcd}$ $\exists B \to X_{n+1}$ s.t.:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "="] & X_n \arrow[d, "p_n"] \\ B \arrow[r, "\exists"'] & X_{n+1} \end{tikzcd}$$

- How to construct such an X_{n+1}?
- let $S = \left\{ \begin{array}{c} A \xrightarrow{f} X_n \\ {\scriptstyle j\in J} \big\downarrow \\ B \end{array} ,\ j \in J,\ A \xrightarrow{f} X_n \right\}$

Then

$$\begin{tikzcd}
A \arrow[r, "i_{(j,f)}"] \arrow[d, "f"'] \arrow[rr, bend left, "f"] & \sum\limits_{(j,f)\in S} A \arrow[r, "\sum\limits_{(j,f)\in S} f"] \arrow[d, "\sum\limits_{(j,f)\in S} j"'] & X_n \arrow[d, "p_n^{n+1}"] \\
B \arrow[r, "i_{(j,f)}"'] \arrow[rr, bend right] & \sum\limits_{(j,f)\in S} B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X_{n+1}
\end{tikzcd}$$

pushout

have canonical extensions.

Then consider

$$\begin{array}{c} A \xrightarrow{f} X^{\#} \\ j \big\downarrow \\ B \end{array}$$

- If f factors as

$$\begin{tikzcd} & X_n \arrow[dr, "p_n"] & \\ A \arrow[ur, "f_n"] \arrow[rr, "f"] & & X^{\#} \end{tikzcd}$$

for some n, then

$$\begin{tikzcd} A \arrow[r, "f_n"] \arrow[d, "j"] \arrow[rr, bend left, "f"] & X_n \arrow[d, "p^n_{n+1}"'] \arrow[r, "p_n"] & X^{\#} \\ B \arrow[r, "\exists"] & X_{n+1} \arrow[ur, "p_{n+1}"'] & \end{tikzcd}$$

so we get desired extension.

& have solved problem.

- Indeed this always works in Mod_R (or any algebraic cat) if A is finitely generated by $a_1, \dots, a_n$:

indeed $f a_i$ factors through $X_{m_i} \xrightarrow{p_{m_i}} X^{\#}$ as these maps are jointly surj, & then take $m = \max(m_1, \dots, m_n)$ we see that f factors through $p_m X_m \longrightarrow X^{\#}$.

- If object A is not fin. gen., need to take a longer chain of cardinality

$|A|$! But we can do that in a cocomplete category like Mod_R.

- For more on proj. & inj. objects come to Algebra IV.