

lecture 13 - Projective & injective modules

Defⁿ) An R -module M is projective if:
 given $f: A \rightarrow B$ surj, & $g: M \rightarrow B$ then
 $\exists g': M \rightarrow A$ such that

$$\begin{array}{ccc} & M & \\ \downarrow g' & & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

Proposition

Each free module is projective.

Proof

let FX be free & consider

$$\begin{array}{ccc} FX & \xrightarrow{g} & B \\ \downarrow \gamma & & \downarrow \gamma \\ A & \xrightarrow{f \text{ surj}} & B \end{array} \quad \text{Then we have } X \xrightarrow{\bar{g}} B \text{ by } F+U.$$

Since uf is surjective function, $\exists s: UB \rightarrow UA$
 s.t. $uf \cdot s = 1$. Therefore

$$t = s \cdot \bar{g} \quad \text{Then by } F+U \text{ again, } \begin{array}{ccc} FX & \xrightarrow{\bar{g}} & B \\ \downarrow \gamma & & \downarrow \gamma \\ UA & \xrightarrow{uf} & UB \end{array} \quad \begin{array}{ccc} FX & \xrightarrow{\bar{g}} & B \\ \downarrow \gamma & & \downarrow \gamma \\ A & \xrightarrow{f} & B \end{array} \quad \square$$

We will prove that projective modules \equiv retracts of free modules.

Defⁿ) In a category \mathcal{C} , we say A is a retract of B if $\exists A \xrightarrow{f} B$ w' $gf = 1_A$.

Theorem

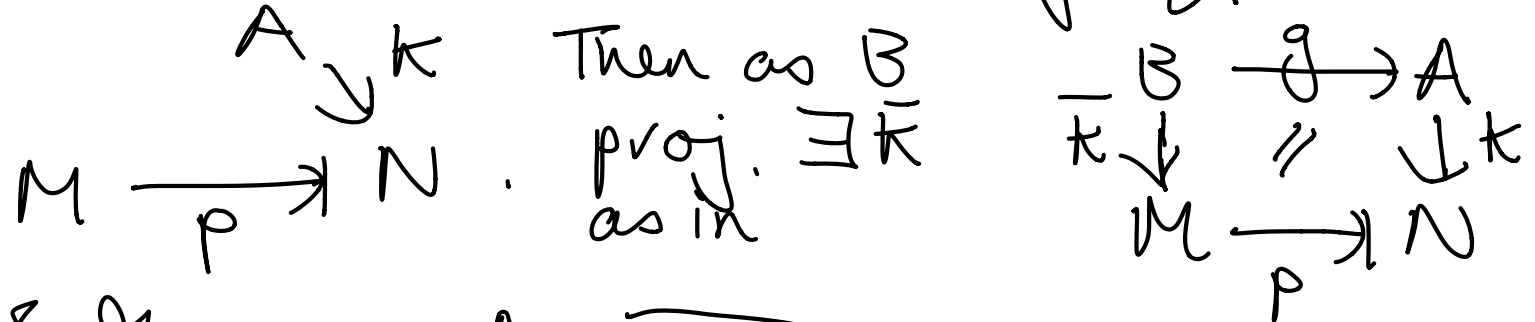
Projective modules \equiv retracts of frees.

Proof

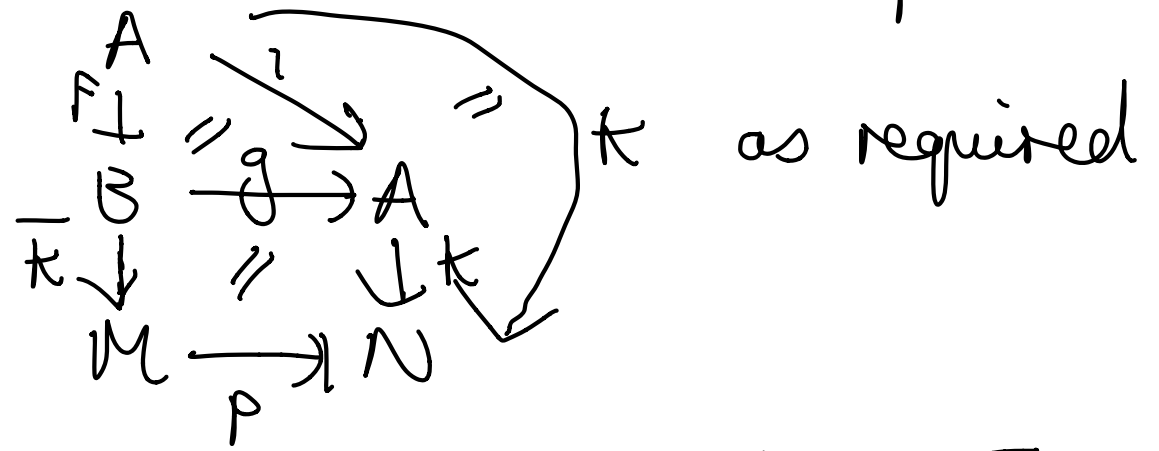
We know free \Rightarrow projective. So for one direction, sufficient to show that retracts of projective modules are proj.

- Consider $A \begin{matrix} \xleftarrow{g} \\ \xrightarrow{f} \end{matrix} B$ w' $gf=1$ & B proj.

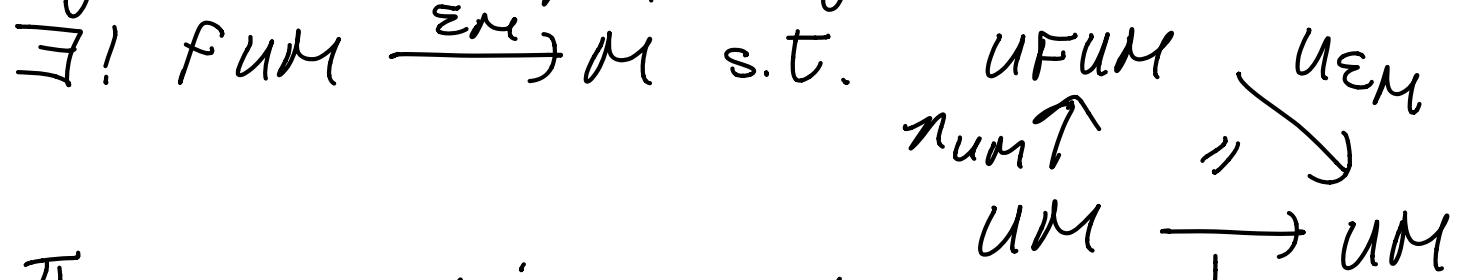
- Consider $M \xrightarrow{p} N$ surj &



& then take



- Conversely, let M be projective. Then by the univ. prop. of FUM,



The same diagram shows ϵ_M is surj.

Therefore as M is projective,

$$\begin{array}{ccc} & M & \\ \exists k \swarrow & \downarrow & \downarrow \\ FUM & \xrightarrow{\varepsilon_M} & M \end{array} \quad \text{s.t. } \varepsilon_M \cdot k = 1$$

so M is a retract of FUM , as required.

Example

Consider ring \mathbb{Z}_6 . Then \mathbb{Z}_6 is free \mathbb{Z}_6 -module &

$$\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \text{ as modules.}$$

Since direct summands are retracts

$$\begin{array}{ccc} a & \longleftarrow & (a, b) \\ \text{eg. } \mathbb{Z}_2 & \longleftarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \\ a & \longleftarrow & (a, 0) \end{array}$$

we see, by prev. theorem, that

$\mathbb{Z}_2, \mathbb{Z}_3$ are projective (though not free.)

Injective modules

- Dual to projective modules.

Defⁿ) A module M is injective if for each mono $i: A \hookrightarrow B$ & $A \xrightarrow{f} M$

$$\exists B \xrightarrow{\bar{f}} M \text{ st } \begin{array}{ccc} A & \xrightarrow{i} & B \\ & f \searrow & \swarrow \bar{f} \\ & M & \end{array}$$

- To understand inj modules in practise, the following is very useful.

Theorem (Baer criterion)

M is injective \Leftrightarrow it has the extension property wvt submodules $I \hookrightarrow R$ (ie. ideals of R)

Proof

- It suff. to show that M has ext. property wvt submodule inclusion $A \hookrightarrow B$.

- Consider $\begin{array}{ccc} A & \hookrightarrow & B \\ & f \searrow & \\ & M & \end{array}$

- Consider pairs (B_i, f_i) where $A \subseteq B_i \subseteq B$
submod. monom.

& $f_i: B_i \rightarrow M$ st $f_i|_A = f$.

- Write $(B_i, f_i) \leq (B_j, f_j)$ if $B_i \subseteq B_j$
& $f_j|_{B_i} = f_i$

$$\left(\begin{array}{c} \text{e. } A \hookrightarrow B_i \hookrightarrow B_j \hookrightarrow B \\ f \searrow \quad \downarrow f_i \quad \downarrow f_j \quad \swarrow \\ \quad \quad \quad M \end{array} \right)$$

- The set of such pairs is a poset.
- Will use Zorn's lemma to show it has a maximal element:

so consider a chain $(B_i, f_i)_{i \in I}$ of submodules of B & maps f_i , & form $\bigcup_{i \in I} B_i = \{x \in B : \exists i \in I, x \in B_i\} \subseteq B$.

Then $\bigcup_{i \in I} B_i$ is submodule, & can define homomorphism $f^*: \bigcup_{i \in I} B_i \rightarrow M$ by

$$x \longmapsto f_i(x) \text{ if } x \in B_i.$$

- This is a well def. hom. extending f . Thus the chain has an upper bound \Rightarrow by Zorn the poset has a max^e element (N, g) as pictured:

$$\begin{array}{ccccc} A & \hookrightarrow & N & \hookrightarrow & B \\ & & \downarrow g & & \\ & & M & & \end{array}$$

- Suppose that $N \neq B$. Will show this leads to a contradiction, compl. the proof.
- Indeed let $b \in B - N$ & consider $I = \{r \in R : rb \in N\} \subseteq R$.

In fact

$$\begin{array}{ccc} I & \hookrightarrow & R \\ \downarrow \cdot b & \dashv & \downarrow \cdot b \\ N & \hookrightarrow & B \end{array} \quad \begin{array}{c} r \\ \downarrow \\ r b \end{array}$$

is a pullback.

- Then have

$$\begin{array}{ccc} I & \hookrightarrow & R \\ \downarrow & & \downarrow \\ N & \hookrightarrow & N + Rb \subseteq B \end{array}$$

(Real idea: this is a pushout!)

the submodule $N + Rb = \{ n + rb : n \in N, r \in R \} \subseteq B$.

- If we can show that

$g: N \rightarrow M$ extends to $N + Rb$, this will also extend f & so contradict maximality of N , completing the proof.

- Consider

$$\begin{array}{ccc} I & \hookrightarrow & R \\ \downarrow \cdot b & & \downarrow \cdot b \\ N & \hookrightarrow & N + Rb \\ \downarrow g & & \\ M & & \end{array}$$

$\exists h$ by assumption

- Define

$$N + Rb \xrightarrow{\ell} M$$

$\ell(n + rb) = gn + hr$
which clearly extends g .

- Must show ℓ is a module map: Firstly, well defined:

- suppose $n + rb = n' + r'b \in N + Rb$.

- Then $(r-r')b = n'-n$, so $r-r' \in I$.

$$\begin{aligned} \text{Then } l(nr+rb) - l(n'+r'b) &= \\ gnr + hr - gn' - hr' &= \\ g(n-n') + h(r-r') &\text{ \& } r-r' \in I \\ = g(n-n') + g((r-r')b) &= 0 \\ \text{as } (r-r')b &= n-n'. \end{aligned}$$

• More easily, l is a homomorph. \square

Corollary

In Ab , A is injective \Leftrightarrow it is divisible: that is,

$$\forall a \in A \text{ \& } n \neq 0 \in \mathbb{N}$$

$$\exists b \in A \text{ st } nb = a.$$

Proof

- Here $R = \mathbb{Z}$ ring of integers.

- Ideals of \mathbb{Z} are principal

$$n\mathbb{Z} \hookrightarrow \mathbb{Z}.$$

Consider

$$\begin{array}{ccc}
 n\mathbb{Z} & \xrightarrow{f} & A \\
 \downarrow & & \\
 \mathbb{Z} & &
 \end{array}$$

- A map f is spec. by $a \in A$ st.
 $f(n) = a$.

- To give an extension $g: \mathbb{Z} \rightarrow A$
of f along $n\mathbb{Z} \hookrightarrow \mathbb{Z}$ is
to give $g(1) = b \in A$ st
 $a = g(n) = \underbrace{g(1) + \dots + g(1)}_{n \text{ times}}$
 $= n \cdot b$

Then by Baer criterion, A is
inj \Leftrightarrow all such extensions
exist - that is, just when
 A is divisible. \square

Projective & injective resolutions

- In homological alg, it is important that each module M has a proj. & inj resolution.

- For this, key points are:

① Given M we can find proj. M' & surjection $M' \rightarrow M$.

② - - - - - inj. M' & injection $M \rightarrow M'$.

Prop ① is easy (Take $F \cup M \xrightarrow{\epsilon_M} M$).

② I will sketch now.

- Special case of so-called small object argument:

we are given some set J of morphisms $j: A \rightarrow B$ in a cat \mathcal{C} , and we want to find for each X , a map

$X \xrightarrow{f} X^\#$ with $X^\#$ J -injective:

given $j: A \longrightarrow B \in \mathcal{J}$

& $A \xrightarrow{f} X^\#$

$j \perp \dashrightarrow$

$B \dashrightarrow \exists.$

Example: For $\text{Mod } R$, we take submodules $I \hookrightarrow R$ & obtain injectivity.

How to construct $X^\#$?

As colimit of a chain

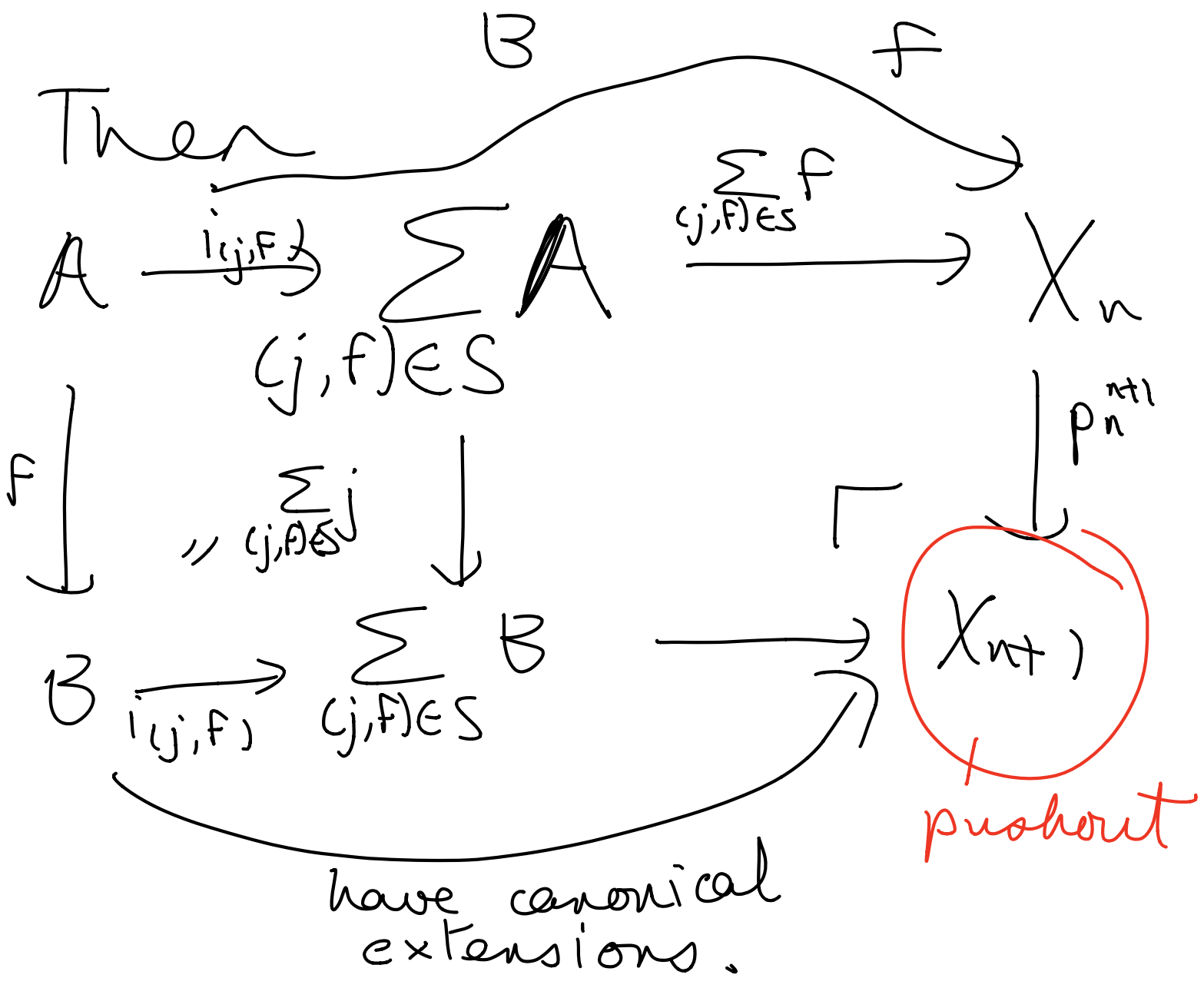
$$\begin{array}{ccccccc}
 X = X_0 & \longrightarrow & X_1 & \longrightarrow & \dots & \longrightarrow & X_n \xrightarrow{P_{n+1}^n} X_{n+1} \longrightarrow \dots \\
 & & & & & & \downarrow P_n \quad \downarrow P_{n+1} \\
 & & & & & & X^\#
 \end{array}$$

$P = p_0$

where $X = X_0$ & where X_{n+1} contains extensions for lifting problems in X_n :
more precisely,

$$\begin{array}{ccc}
 \exists A \xrightarrow{f} X_n & \exists B \rightarrow X_{n+1} & A \xrightarrow{f} X_n \\
 j \in \mathcal{J} \perp & \text{s.t.} & \downarrow \text{"} \\
 B & & B \xrightarrow{\exists} X_{n+1} \\
 & & \perp P_n
 \end{array}$$

- How to construct such an X_{n+1} ?
- let $S = \left\{ \begin{array}{c} A \xrightarrow{f} X_n, j \in J, A \xrightarrow{f} X_n \\ \downarrow \\ B \end{array} \right\}$



Then consider

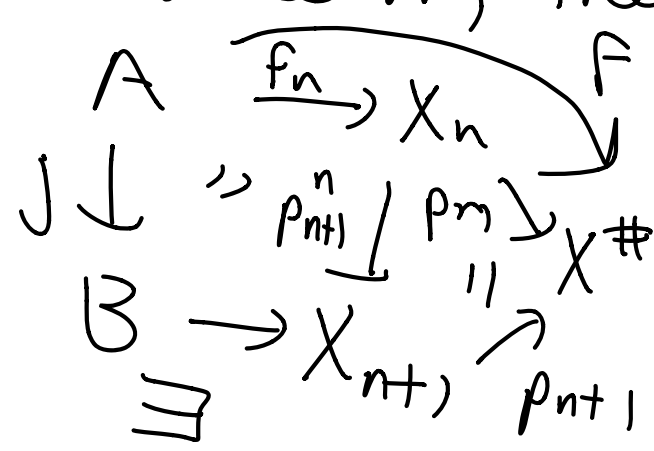
$$A \xrightarrow{f} X^\#$$

$j \perp$
 B

- If f factors as

$$A \xrightarrow{f_n} X_n \xrightarrow{p_n} X^\#$$

For some n , then



so we get
desired
extension.

& have solved problem.

- Indeed this always works in $\text{Mod } R$ (or any algebraic cat) if A is finitely generated by a_1, \dots, a_n :

indeed $f a_i$ factors through $X_{m_i}; p_{m_i} \rightarrow X^\#$ as these maps are jointly surj, & then take $m = \max(m_1, \dots, m_n)$ we see that f factors through $p_m X_m \rightarrow X^\#$.

- If object A is not fin. gen., need to take a longer chain of cardinality

$|A|!$ But we can do that in a cocomplete category like $\text{Mod } R$.

- For more on proj. & inj. objects come to Algebra IV.