

Functors

Defⁿ) Let A, B be categories. A functor

$F: A \rightarrow B$ consists of:

- a function $\text{ob } A \rightarrow \text{ob } B$ which we write as $x \mapsto Fx$,
- For each pair $x, y \in A$ a function $A(x, y) \rightarrow B(Fx, Fy)$ written as $x \xrightarrow{\alpha} y \mapsto Fx \xrightarrow{F\alpha} Fy$.

This satisfies the axioms:

- given $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \in A$, we have $F(\beta \circ \alpha) = F\beta \circ F\alpha$.
- given $x \in A$ we have $F(1_x) = 1_{Fx}$.

In other words, a functor sends obs to obs, arrows to arrows (respecting their domain & codomain) & preserves identities and composition.

Examples

- Forgetful Functors:

$$\text{Eg. } U: \text{Grp} \longrightarrow \text{Set}$$
$$(X, m, e) \longmapsto X$$

which sends a group to its underlying set, i.e. forgets the group structure.

Sim. it sends a group homomorphism to its underlying function:

$$f: (X, m_X, e_X) \rightarrow (Y, m_Y, e_Y) \longmapsto f: X \rightarrow Y$$

Clearly preserves composition & identities.

- Similarly, there are forgetful functors from cats of "sets with structure"

To Set, eg
 $U: \text{Top} \rightarrow \text{Set}, U: K\text{-Vect} \rightarrow \text{Set}$
 or Rng, Mon etc...
 Similarly, $U: \text{Rng} \rightarrow \text{Mon}$
 $(R, +, \cdot, 0, 1) \mapsto (R, \cdot, 1)$.

There is a functor $F: \text{Set} \rightarrow \text{Mon}$

where $F X =$ "list/free" monoid on X

- Its elements are (possibly empty) lists $[x_1, x_2, \dots, x_n]$ of elements of X with unit $[-]$ empty list.
- Composition of lists is "concatenation":
 $[x_1, \dots, x_n] \cdot [y_1, \dots, y_m] = [x_1, \dots, x_n, y_1, \dots, y_m]$
- This makes $F X$ a monoid.
- Given a function $f: X \rightarrow Y$ obtain a monoid homomorphism
 $f F: F X \rightarrow F Y$

$[x_1, \dots, x_n] \mapsto [f x_1, \dots, f x_n]$
 & easy to check that this satisfies axioms for a functor.

Example) Let G be a group & consider its suspension ΣG as a groupoid with one object

& $\Sigma G(-, -) = G$.

Then a functor $\Sigma G \rightarrow \text{Set}$ is what?

The functor axioms
 say $e \cdot x = x$ & $h \cdot (g \cdot x) = (h \cdot g) \cdot x$.

$$\begin{array}{ccc}
 g \downarrow & \longmapsto & \begin{array}{ccc} X & & x \\ \downarrow g \cdot & & \downarrow \\ X & & g \cdot x \end{array}
 \end{array}$$

Therefore a functor $\Sigma G \longrightarrow \text{Set}$
 is exactly a G-set: a set X with
 an action of the group G .
 What is a functor $\Sigma G \longrightarrow \text{Vect}$?

Example let Top_* be category of
 pointed spaces (X, a)

& continuous maps which
 preserve basepoint.

There is a functor $\Pi_1 : \text{Top}_* \longrightarrow \text{Grp}$
 where $\Pi_1(X, a) =$ fundamental
 group of loops around a (up to homotopy)

eg.



homotopic
 loops around
 $a \in$
 rectangle

$$\Pi_1(S^1, *) = \mathbb{Z}$$

There are also functors
 $H_n : \text{Top} \longrightarrow \text{Ab} \sim$ abelian group
 sending a top. space
 to its n 'th homological group.

Example : CAT

- Given functors $A \xrightarrow{F} B \xrightarrow{G} C$
we can compose them in obvious way:

$$\begin{array}{ccc} & & C \\ & & \nearrow \\ & G \circ F & \\ & \searrow & \\ & & \end{array}$$

$$x \mapsto G(F(x)) = GFx$$
$$x \xrightarrow{\alpha} y \mapsto GFx \xrightarrow{GF\alpha} GFy$$

- Likewise, we have identity functor $1_A: A \rightarrow A: x \mapsto x$
- In this way, we obtain a (large) category CAT of categories & functors. In fact, this is a 2-category!

Limits & Colimits

Terminal & Initial Objects

Defⁿ) An object $T \in \mathcal{C}$ is terminal if given any other $X \in \mathcal{C}$ there exists a unique morphism

$$X \xrightarrow{f} T$$

Theorem Terminal objects are unique up to unique isomorphism.

Proof) Let S, T be terminal.

As T is terminal, $\exists! S \xrightarrow{f} T$.

As S is terminal, $\exists! T \xrightarrow{g} S$.

As T is terminal, the two maps

$$T \xrightarrow{fg} T \quad \text{are equal.}$$
$$T \xrightarrow{1_T} T$$

As S is terminal, similarly $gf = 1_S$, so that $f: S \rightarrow T$ is an isomorphism.

Example) - In Set , a set is terminal precisely if it has one element,

eg. $\{*\}$

- Similarly, in Grp , Mon , Rng , Top ... the 1-elt set has the unique str of a group, monoid, ring, space... & this makes it the terminal object

in the corresponding category.

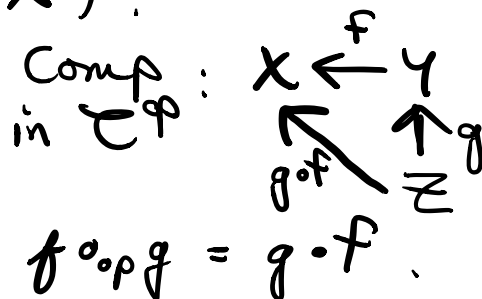
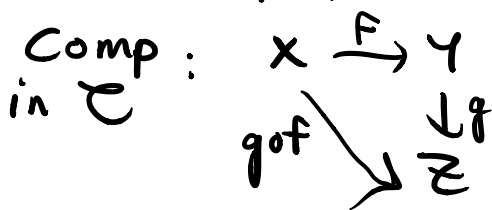
Defⁿ) An object $I \in \mathcal{C}$ is initial if given $X \in \mathcal{C}$ there exists a unique morphism $I \xrightarrow{f} X$.

Remark) Defⁿ of initial object is dual to that of terminal object - all arrows reversed

One can prove that initial objects are unique up to iso. by reversing arrows in proof for terminal objects.

Another approach - use dual/opposite category \mathcal{C}^{op} .

Defⁿ) \mathcal{C}^{op} has same obs as \mathcal{C} & $\mathcal{C}^{\text{op}}(X, Y) = \mathcal{C}(Y, X)$.



$$f \circ p g = g \cdot f$$

Identities in \mathcal{C}^{op} are as in \mathcal{C} .

Def of initial object 2

An ob $I \in \mathcal{C}$ is initial if it is terminal in \mathcal{C}^{op} .

- In this way, we see that init.

objs are unique up to iso Follows
(indeed, is equivalent to) the corr.
statement for terminal objects,
since init objects are term. objs
in \mathcal{C}^{op} .

Principle of
duality

Every categorical defⁿ,
theorem & proof has a
dual, obtained by reversing
the arrows. More formally,

by interpreting original defⁿ, thm.
or proof in \mathcal{C}^{op} instead of \mathcal{C} .

Examples

In Set , the init. object
is the empty set \emptyset since
 $\exists! \emptyset \longrightarrow X$.

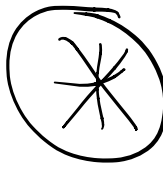
Exercises: What is the initial object
in Grp ? In Rng ?

Products & coproducts

Defⁿ) Let $A, B \in \mathcal{C}$. We say that a diagram

$$\begin{array}{c} P \xrightarrow{p} A \\ \searrow q \\ B \end{array}$$
 is a product diagram

if it has the "universal property" that:
given any other diagram

$$\exists ! X \xrightarrow{h} P \text{ s.t. } \begin{array}{c} X \xrightarrow{h} P \xrightarrow{p} A \\ \searrow q \searrow q \\ B \end{array} \quad \begin{array}{c} X \xrightarrow{f} A \\ \searrow g \\ B \end{array}$$


- In this case, we call the object P the product of A & B , but this is abuse of notation as it is essential to consider the maps p, q .

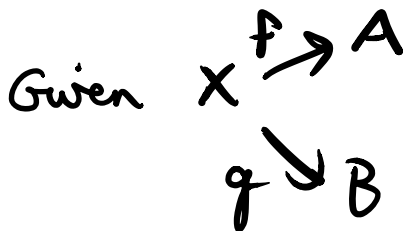
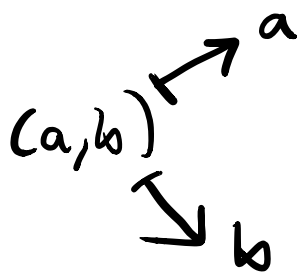
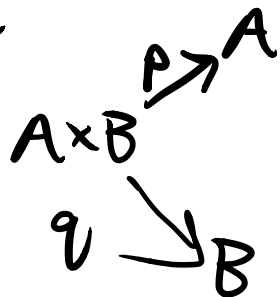
Exercise: The product of A & B is unique up to unique isomorphism.
(Similar to case of terminal objects.)

For this reason, often write product as $A \times B \xrightarrow{p} A$ & call it "the product".

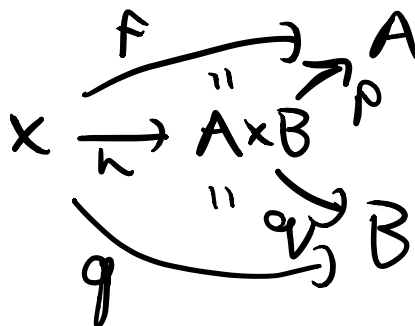
Examples: In Set, the product is the cartesian product

$$A \times B = \{ (a, b) : a \in A, b \in B \}$$

with



if



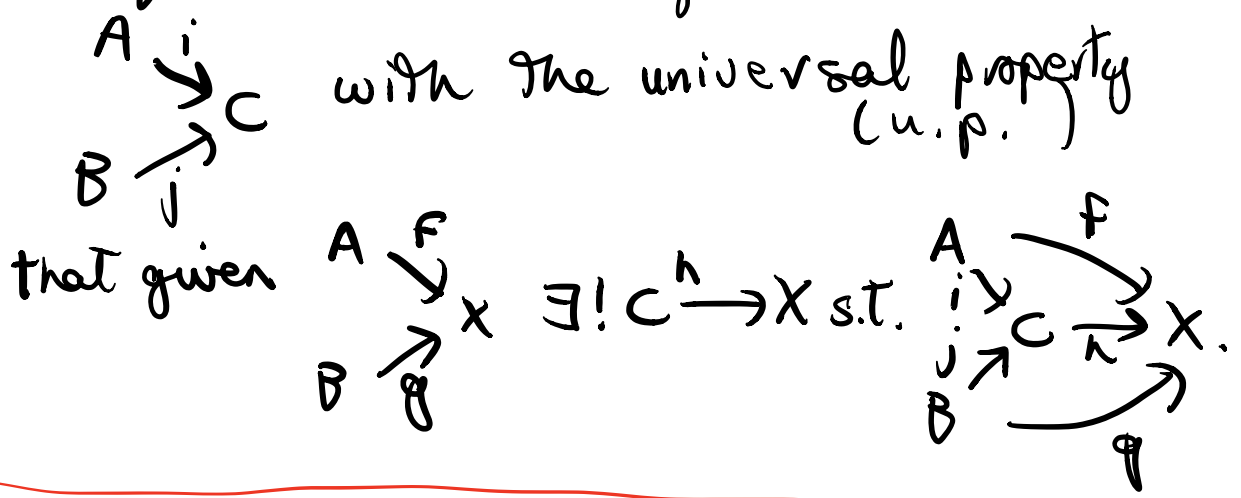
then $phx = fx$ & $qh_x = gx$ so

$$hx = (fx, gx) \text{ . In partic,}$$

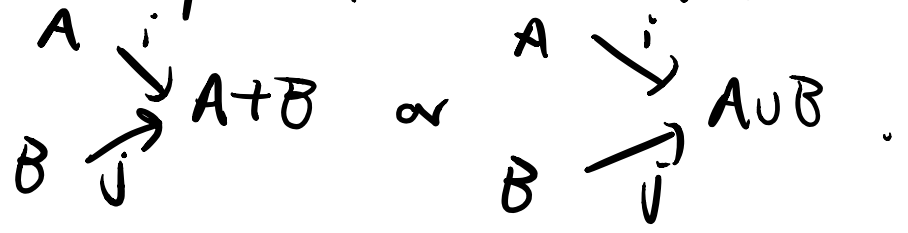
$h : X \rightarrow A \times B : x \mapsto (fx, gx)$ is the unique function s.t. $ph = f$ & $qh = g$ as in $*$.

Similarly, in Grp, Rng etc, products are the usual cartesian products. (Also in Top.)

- The dual notion is that of coproduct: coproducts in \mathcal{C} are products in \mathcal{C}^{op} .
- In elementary terms, the coproduct of A & B is a diagram

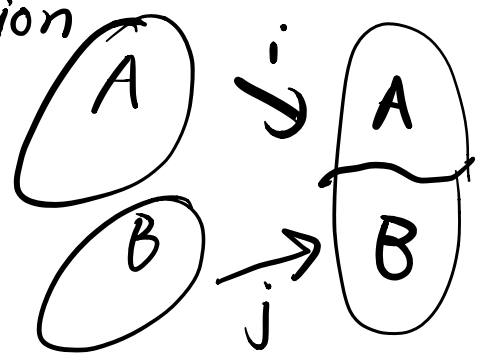


- Again, the coproduct is unique up to unique isomorphism. We often write it as



Examples

- In Set, the coproduct is the disjoint union



$$\begin{aligned}
 A+B &= \\
 &\{ (a,1) : a \in A \} \\
 &\cup \{ (b,2) : b \in B \} \\
 &\text{or various other ways.}
 \end{aligned}$$

- In Grp, the coproduct is the free product:
elts of $G+H$ are "alternating words"
in G & H . eg. $g_1 h_1 g_2 h_2 g_3$

Exercise: Coproducts & products are dual concepts. Thinking about Set, can you see a sense in which addition and multiplication of numbers are dual?

Exercise: What are coproducts of vector spaces, abelian groups or Top. space?

we will look at coproducts in this setting when we study modules.