Lecture 4

Monos & epis

Can we capture injectivity & surjectivity of morphisms in categories?

Def^n) A morphism $f : X \to Y$ is **mono**/monic/a monomorphism if $\forall\, Z \underset{h}{\overset{g}{\rightrightarrows}} X$ satisfying $fg = fh$ then $g = h$.

Dually, $f : X \to Y$ is **epi** if $\forall\, Y \underset{h}{\overset{g}{\rightrightarrows}} Z$ satisfying $gf = hf$ we have $g = h$.

Example

- In Set, monos $\equiv$ injective functions
- Indeed, if f is injective, then
  $fg = fh \Rightarrow fg(z) = fh(z)\ \forall z \in Z$
  $\underset{\text{as } f \text{ inj.}}{\Rightarrow} g(z) = h(z)\ \forall z \Rightarrow g = h$, so f is mono.

Conversely, suppose f is mono: we must show that $\forall x, y \in X$ $fx = fy \Rightarrow x = y$.

Consider 1 element set $1 = (\bullet)$.

We have $(\cdot) \underset{\bar{y}}{\overset{\bar{x}}{\rightrightarrows}} X \xrightarrow{f} Y$

let $\bar{x} : (\cdot) \longrightarrow X : \cdot \longmapsto x$

$\bar{y} : (\cdot) \longrightarrow X : \cdot \longmapsto y$.

& then $f \circ \bar{x} = f \circ \bar{y}$ since at unique element $\bullet$ we have

$$\underset{f x}{\underset{\shortparallel}{f \circ \bar{x}(\cdot)}} = \underset{f y}{\underset{\shortparallel}{f \circ \bar{y}(\cdot)}}$$

so as f is mono we have $\bar{x} = \bar{y}$,

so $\underset{x}{\underset{\shortparallel}{\bar{x}(\cdot)}} = \underset{y}{\underset{\shortparallel}{\bar{y}(\cdot)}}$.

Idea: notion of monomorphism in a cat. abstracts injections in Set by replacing 1-element set by a general object.

whilst in Set, $1 \longrightarrow X$ is an "element" of X, in a gen. cat we think of morphisms $A \longrightarrow X$ as generalised elements.

Ex) In all algebraic categories, mono $\equiv$ injective homomorphisms

Proof that injective $\Rightarrow$ mono is just as in Set; we will prove mono $\Rightarrow$ injective in section on

universal algebra.

Example

- In Set, epi $\equiv$ surjective functions.
  - Surjective $\Rightarrow$ epi is easy (similar to injective $\Rightarrow$ mono)

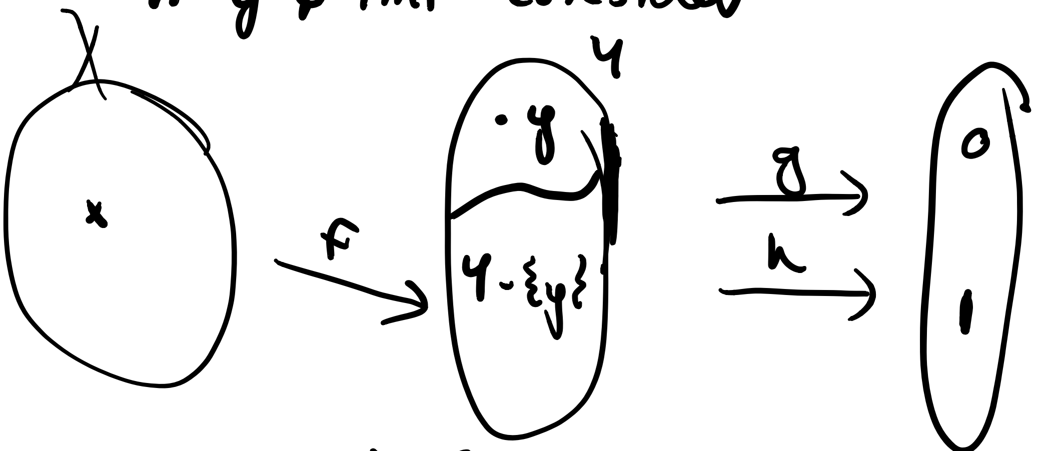
ie. if $X \xrightarrow{f} Y \overset{g}{\underset{h}{\rightrightarrows}} Z$ sat $gf = hf$

then let $y \in Y$. As f is surj.,
$\exists x \in X$ such that $fx = y$.

Hence $\underset{\overset{\|}{gy}}{gfx} = \underset{\overset{\|}{hy}}{hfx}$ for all y

so $g = h$.

- Conversely, let $X \xrightarrow{f} Y$ be epi & consider $y \in Y$.
  If $y \notin \operatorname{im} f$ consider



where $gy = 0$ & $gz = 1$ otherwise
& $hz = 1$ all $z \in Y$.

Then $gfx = 1 = hfx$ all $x \Rightarrow g = h$ (as f is epi)

but this is false as $gy \neq hy$.
Hence we conclude $y \in \operatorname{im} f$
$\Rightarrow$ f is surjective.

- In algebraic categories,
  surjective $\Rightarrow$ epi (just as above)

but the converse is not true in general.

In fact, in Rng, the inclusion homomorphism

$$\mathbb{Z} \hookrightarrow \mathbb{Q}$$

is epi but not surjective.

Exercise: check the above!

However,

surjective $\equiv$ regular "epi"

which we will look at this when we study congruences & quotients in universal algebra.

Natural Transformations

Def^n) Let $F, G : A \rightrightarrows B$ be functors.

A natural transformation $\eta : F \Rightarrow G$ consists of :

- For each $x \in A$ a morphism $\eta_X : FX \longrightarrow GX$ such that :
- For all $\alpha : X \longrightarrow Y \in A$ the square

$$\begin{tikzcd}
FX \arrow[r, "\eta_X"] \arrow[d, "F\alpha"'] & GX \arrow[d, "G\alpha"] \\
FY \arrow[r, "\eta_Y"'] & GY
\end{tikzcd}$$

commutes.

Remark : We write

$$\begin{tikzcd}
A \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\eta"']
\end{tikzcd}$$

for a natural transformation.

Examples

① Let $G = \boxed{1 \overset{s}{\underset{t}{\rightrightarrows}} 0}$. A diagram $X : G \longrightarrow \mathcal{C}$ consists of objects & arrows

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "Xs = s_X"] \arrow[r, shift right, "Xt = t_X"'] & X_0
\end{tikzcd}$$

& a nat. t. $\eta : X \rightarrow Y$ consists of

$$\begin{tikzcd}
X_1 \arrow[r, "\eta_1"] \arrow[d, shift right, "s_X"'] \arrow[d, shift left, "t_X"] & Y_1 \arrow[d, shift right, "s_Y"'] \arrow[d, shift left, "t_Y"] \\
X_0 \arrow[r, "\eta_0"] & Y_0
\end{tikzcd}$$

making both squares commutes.

- When $\mathcal{C} = \mathrm{Set}$, a diag X as above, consists of a set X_0 of "vertices" $x, y, z \dots$
  - a set X_1 of "directed edges" f, g, h which have "source" $s_X(f) \in X_0$ & target $t_X(f) \in X_0$

& we write $a \xrightarrow{f} b$ if $s_X(f) = a$ & $t_X(f) = b$.

- In this way, we see that a diag. $X : G \longrightarrow \mathrm{Set}$ is exactly a directed graph eg.

- A natural transf.

$$\begin{tikzcd}
X_1 \arrow[r, "\eta_1"] \arrow[d, shift right, "s_X"'] \arrow[d, shift left, "t_X"] & Y_1 \arrow[d, shift right, "s_Y"'] \arrow[d, shift left, "t_Y"] \\
X_0 \arrow[r, "\eta_0"] & Y_0
\end{tikzcd}$$

: edges in $X \mapsto$ edges in Y

& vertices in $X \mapsto$ vertices in Y

in a way that preserves source & target:

i.e. $x \longmapsto \eta_0 x$

$x \xrightarrow{\alpha} y \longmapsto \eta_0 x \xrightarrow{\eta_1 \alpha} \eta_0 y$

i.e. if $x = s_X(\alpha)$ then

$\eta_0 x = \eta_0 s_X(\alpha) = s_Y \eta_1(\alpha)$,

& sim. for t.

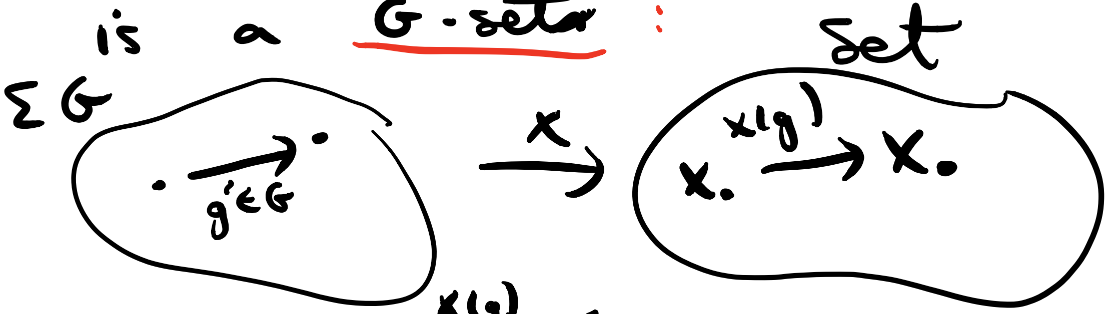
So a natural transformation is precisely a morphism of directed graphs.

Ex 2

- If A, B are posets viewed as categories where $\exists!\, a \to b \iff a \leq b$
- Given $F, G : A \rightrightarrows B$ are order-preserving functions viewed as functors then a natural transformation $\eta : F \Longrightarrow G$ means $Fx \leq Gx \ \forall x$.
- The commutativity condition is redundant since all diagrams commute in B.

Ex 3

If G a group, we have 1-object category ΣG. A functor $\Sigma G \xrightarrow{X} \mathrm{Set}$ is a G-sets :



where $X_\cdot \xrightarrow{X(g)} X_\cdot : x \mapsto g \cdot x$.

What is a natural transformation?

Ex 4

word/list monoid ~ forgetful

Consider
$$\begin{tikzcd} \mathrm{Set} \arrow[r, "F"] \arrow[rr, bend right, "W"'] & \mathrm{Mon} \arrow[d, phantom, "\shortparallel"] \arrow[r, "U"] & \mathrm{Set} \\ & {} & \end{tikzcd}$$

$X \longmapsto WX =$ set of words/lists $[x_1, \ldots, x_n]$ in X

We can define a natural transformation $\eta : 1_{set} \Longrightarrow W$ whose component at X is map $X \xrightarrow{\eta_X} WX : x \longmapsto [x]$ ~ word of length 1

At $f : X \to Y$ we need

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & WX \arrow[d, "Wf"] \\ Y \arrow[r, "\eta_Y"'] & WY \end{tikzcd}$$

By def^n, $Wf[x_1, \ldots, x_n] = [fx_1, \ldots, fx_n]$.

so $Wf\eta_x[x] = Wf[x] = [fx]$
$= \eta_Y(fx) = \eta_Y \circ f(x)$.

Functor categories

- Consider categories $\mathcal{A}$ & $\mathcal{B}$.
- Given natural transformations

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{B} \arrow[from=F, to=G, Rightarrow, shorten=2pt, "\alpha"'] \arrow[from=G, to=H, Rightarrow, shorten=2pt, "\beta"'] \end{tikzcd}$$

we can compose them (vertically) to obtain a natural tr.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "H"'{name=H}] & \mathcal{B} \arrow[from=F, to=H, Rightarrow, shorten=2pt, "\beta.\alpha"'] \end{tikzcd}$$

with components

$$FX \xrightarrow{\alpha_X} GX \xrightarrow{\beta_X} HX \text{ for each } X.$$

The "naturality cond." is easy to check, & this composition of natural transformations is associative (as composition in $\mathcal{B}$ is associative)

- Also have identity nat. transf.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "F"'{name=F2}] & \mathcal{B} \arrow[from=F, to=F2, Rightarrow, shorten=2pt, "1_F"'] \end{tikzcd}$$

with components

$$1_{FX} : FX \longrightarrow FX$$

- Altogether, we obtain a category $[\mathcal{A},\mathcal{B}]$ called the functor category:

objects - functors $A \to B$
arrows - natural transformations.

Example

- Functor category $[1 \rightrightarrows 0, \mathrm{Set}]$ is the category of directed graphs & graph homomorphisms.
- For G a group, what is $[\Sigma G, \mathrm{Set}]$? Or $[\Sigma G, \mathrm{Vect}]$?

Horizontal composition

- Given

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\eta \Downarrow"] & B \arrow[r, "H"] & C \end{tikzcd}$$

we can define

$$\begin{tikzcd} A \arrow[r, bend left, "HF"] \arrow[r, bend right, "HG"'] \arrow[r, phantom, "H\eta \Downarrow"] & C \end{tikzcd}$$

to be the nat. transf. with components: at $x \in A$, $HFX \xrightarrow{H\eta_x} HGX$.

- Given

$$\begin{tikzcd} A \arrow[r, "F"] & B \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\eta \Downarrow"] & C \end{tikzcd}$$

we define

$$\begin{tikzcd} A \arrow[r, bend left, "GF"] \arrow[r, bend right, "HF"'] \arrow[r, phantom, "\eta_F \Downarrow"] & C \end{tikzcd}$$

as the natural transf. with component

$\eta_{FX} : GFX \longrightarrow HFX$ at $x \in \mathcal{A}$.

Given
$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "F", ""{name=U, below}] \arrow[r, bend right=50, "G"', ""{name=D}] & \mathcal{B} \arrow[r, bend left=50, "H", ""{name=U2, below}] \arrow[r, bend right=50, "I"', ""{name=D2}] & \mathcal{C}
\arrow[Rightarrow, from=U, to=D, "\alpha"']
\arrow[Rightarrow, from=U2, to=D2, "\beta"']
\end{tikzcd}$$

we have two ways of defining a composite, as either path in

$$\begin{tikzcd}
HF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "H\alpha"'] & IF \arrow[d, Rightarrow, "I\alpha"] \\
HG \arrow[r, Rightarrow, "\beta G"'] & IG
\end{tikzcd}$$
(i.e. $I\alpha . \beta_F$) ;

these agree by naturality of

$$\begin{tikzcd}
\beta : & HFX \arrow[r, "\beta_{FX}"] \arrow[d, "H\alpha_X"'] & IFX \arrow[d, "I\alpha_X"] \\
& HGX \arrow[r, "\beta_{GX}"'] & IGX
\end{tikzcd}$$

at the morphism $\alpha_X : FX \to GX$.

The resulting nat t.
$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "HF", ""{name=U, below}] \arrow[r, bend right=50, "IG"', ""{name=D}] & \mathcal{C}
\arrow[Rightarrow, from=U, to=D, "\beta * \alpha"']
\end{tikzcd}$$

is called horizontal composite.

Remark Categories, Functors & natural transformations form a 2-category!

Equivalence of categories

- When are two categories "the same"?
- CAT is a category, so we can speak of isos of cats., but this is too strong a notion.
- Better notion: equivalence of cats.

Def^n) A natural transformation

$$\begin{tikzcd} A \arrow[r, bend left=60, "F"{name=U}] \arrow[r, bend right=60, "G"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\eta"'] \end{tikzcd}$$

is a natural isomorphism if it is an isomorphism in $[A, B]$.

We write $\eta : F \overset{\sim}{\Rightarrow} G$ for a nat. isomorphism.

Lemma $\eta : F \Rightarrow G$ is a nat. iso. $\iff$ $\eta_X : FX \longrightarrow GX$ is an isomorphism in B.

Proof Exercise.

Def^n A functor $F : A \longrightarrow B$ is an equivalence of categories if $\exists$ functor $G : B \to A$ and natural isos

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[dr, bend right, "1_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "G"] \arrow[dr, bend left, "1_{\mathcal{B}}"] \arrow[dl, phantom, "\alpha \ \cong"] \arrow[dr, phantom, "\beta \ \cong"'] & \\
& \mathcal{A} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}$$

.

Example:

Let FinSet be cat. of finite sets & functions, $\mathcal{F}$ the cat cont. the sets

$\emptyset, \{0\}, \{0,1\}, \{0,1,2\}, \ldots$

$\{0,1,2,\ldots,n\}$ for all $n \in \mathbb{N}$

& functions between them.

Show $\mathcal{F} \xhookrightarrow{\text{inclusion}} \text{FinSet}$

is an equivalence of categories.