Lecture 5 - Adjoint Functors

- Key concepts in cat. theory: adjunctions

Def^n) An adjunction consists of functors

$$A \underset{U}{\overset{F}{\rightleftarrows}} B$$

together with bijections

$$B(Fa, b) \xrightarrow{\varphi_{a,b}} A(a, Ub)$$

for each $a \in A$, $b \in B$ and these are natural in each variable:

- naturality in b means: given $Fa \xrightarrow{\alpha} b$ & $b \xrightarrow{\beta} b'$ we have $\varphi_{a,b'}(\beta \circ \alpha) = U\beta \circ \varphi_{a,b}(\alpha)$
- naturality in a means: given $Fa \xrightarrow{\alpha} b$ & $a' \xrightarrow{\beta} a$ we have $\varphi_{a',b}(\alpha \circ F\beta) = \varphi_{a,b}(\alpha) \circ \beta$.

Remark: **Key point** is that if we an adjunction,

$$\frac{\text{maps } Fa \longrightarrow b}{\text{maps } a \longrightarrow Ub}$$

bijectively corresp. to maps $a \longrightarrow Ub$ in a natural way.

- Naturality above may seem a little obscure, but it just means two functors are nat. isomorphic.
- In the examples below, we won't check naturality conds - you should do it!

Notation : We say that F is left adjoint to U, & write $F \dashv U$.

Examples

Free $\dashv$ forgetful !

a) consider

$F \dashv U$

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, shift right, "U \text{ forgetful}"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right, "F \text{ free vector sp.}"'] \end{tikzcd}$$

- $FX = \{ \lambda_1 x_1 + \dots + \lambda_n x_n : x_1, \dots, x_n \in X, \lambda_1, \dots, \lambda_n \in k \}$

  the vector space with basis X

- At a function $f : X \to Y$, $Ff : FX \longrightarrow FY : \sum_i \lambda_i x_i \longmapsto \sum_i \lambda_i f(x_i)$ is obtained by linear 'extension'.

- We have a bijection

$$\begin{aligned} \mathrm{Vect}(FX, Y) &\longrightarrow \mathrm{Set}(X, UY) \\ FX \xrightarrow{f} Y &\longmapsto X \longrightarrow UY, \\ & \qquad\; x \longmapsto f(x)) \end{aligned}$$

- This is a bij^n, whose inverse sends $g : X \to UY$ to the linear map $FX \longrightarrow Y$:

$$\sum \lambda_i x_i \longmapsto \sum \lambda_i g(x_i).$$

- why inverse parts of a bij^n? since $f(\sum_i \lambda_i x_i) = \sum_i \lambda_i f(x_i)$ by linearity.

b) Similarly, we have an adjunction

$$\begin{tikzcd} \mathrm{Mon} \arrow[r, shift right=1.5ex, "U = \text{Forgetful}"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right=1.5ex, "F = \text{Free}"'] \end{tikzcd}$$

- Recall FX = list monoid .. $[x_1, \dots, x_n] \in FX$
- Given $FX \xrightarrow{f} Y$, corresponding map
  $X \longrightarrow UY : x \longmapsto \underbrace{f[x]}_{\text{word of length 1.}}$
- Given $X \xrightarrow{g} UY$, the corresponding map
  $\bar{g} : FX \longrightarrow Y$:
  $[x_1, \dots, x_n] \longmapsto g(x_1) \cdot \dots\dots \cdot g(x_n)$
  where $\cdot$ denotes multiplication in monoid Y.
- Note that this defⁿ is forced on us since $[x_1, \dots, x_n] = [x_1] \cdot \dots \cdot [x_n]$ & $\bar{g}$ must preserve multiplication & sat. $\bar{g}[x] = g(x)$ for $x \in X$ in order to have a bijⁿ.

c) Similarly, the forgetful functors from Grp or Rng to Set have left adjoints, sending a set to free group/free ring ...
More generally, forgetful

Functors in universal algebra have left adjoints - we will see this soon.

Other examples

- The forgetful functor $U : \mathrm{Grp} \longrightarrow \mathrm{Mon}$ has a left adjoint, & a right adjoint R: this sends M to subgroup RM of invertible elements in M

If G a group, M a monoid, then a monoid map $U(G) \longrightarrow M$ takes elements of G to invertible elements of M, & so factors as $G \longrightarrow R(M)$ through $R(M) \hookrightarrow M$.

We obtain $\dfrac{U(G) \longrightarrow M}{G \longrightarrow R(M)}$ a bij^n.

- We have adjoint functors

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, "\perp\ U"] & \mathrm{Set} \arrow[l, shift right=4ex, "D = \text{discrete}"'] \arrow[l, shift left=4ex, "\perp"', "I = \text{indiscrete}"]
\end{tikzcd}$$

DX = set X with all subsets open.
IX = set X with $X, \emptyset$ open.
Then any function $X \to UY$ is cts wrt discrete topology so

$$\frac{X \to UY}{DX \to Y} \quad \text{bij}^n.$$

Sim. any function $UX \to Y$ is cts wrt indiscrete top. on Y

so $$\frac{UX \to Y}{X \to IX}.$$

- Let $A \in \mathrm{Set}$. We have a functor

$$\begin{aligned} A \times - : \mathrm{Set} &\longrightarrow \mathrm{Set} \\ X &\longmapsto A \times X. \end{aligned}$$

Now there is a bij^n between functions $A \times X \xrightarrow{f} Y$

& Functions $A \to Y$

Functions $X \xrightarrow{\overline{F}} \mathrm{Set}(A, Y) = Y^A$

where $\overline{F}_x : A \longrightarrow Y$

$$a \longmapsto F(x, a)$$

This process is called $\underline{\text{currying}}$.

Therefore we have adjunction

$\underline{A \times - \dashv (-)^A}$ where

$$\begin{aligned} (-)^A : \mathrm{Set} &\longrightarrow \mathrm{Set} \\ X &\longmapsto X^A . \end{aligned}$$

Properties of adjunctions

- Consider an adjunction $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "U"] \end{tikzcd}$ with nat. bijections $\mathcal{B}(Fa,b) \xrightarrow[\varphi_{a,b}]{} \mathcal{A}(a,Ub)$.

- Taking $b = Fa$, we obtain

$$\begin{tikzcd} \mathcal{B}(Fa,Fa) \arrow[r, "\varphi_{a,Fa}"] & \mathcal{A}(a,UFa) \\ Fa \xrightarrow{1} Fa \arrow[r, mapsto] & a \xrightarrow{\eta_a} UFa \end{tikzcd}$$

which is called the unit of the adjunction.

Lemma) These components give a natural transformation $\eta : 1 \Longrightarrow UF$ (called the unit).

Proof) We have

$$\begin{tikzcd} a \arrow[r, "\eta_a = \varphi_{a,Fa}(1)"] \arrow[d, "\alpha"'] \arrow[dr, "\varphi_{a,Fb}(F\alpha)", "②"'] & UFa \arrow[d, "UF\alpha", "①"'] \\ b \arrow[r, "\eta_b = \varphi_{b,Fb}(1)"'] & UFb \end{tikzcd}$$

- Triangle ① commutes using naturality of φ at

$$\begin{tikzcd} Fa \arrow[r, "1"] \arrow[dr, "F\alpha"'] \arrow[dr, phantom, bend left, "="] & Fa \arrow[d, "F\alpha"] \\ & Fb \end{tikzcd}.$$

- Triangle ② commutes using nat of φ in first variable at

$$\begin{tikzcd} Fa \arrow[d, "F\alpha"'] \arrow[dr, "F\alpha"] \arrow[dr, phantom, bend right, "="] & \\ Fb \arrow[r, "1"'] & Fb \end{tikzcd}.$$

Therefore the square commutes & $\eta : 1 \Rightarrow UF$ is a natural transformation. ◻

Examples of the unit

- Consider $$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Vect} \arrow[l, shift left=1.5, "U"] \end{tikzcd}$$.

- $FX \xrightarrow{1} FX \overset{\varphi}{\longmapsto} X \xrightarrow{\eta_x} UFX$
  $x \longmapsto x$.

  So η_x is standard inclusion of basis elements.

- For $$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Mon} \arrow[l, shift left=1.5, "U"] \end{tikzcd}$$,

  $X \xrightarrow{\eta_x} UFX : x \longmapsto [x]$.

In fact, the components

$$\varphi_{a,b} : B(Fa, b) \longrightarrow A(a, Ub)$$

are determined by η:

given $\alpha : Fa \to b$, naturality at

$$\begin{tikzcd} Fa \arrow[r, "1"] \arrow[dr, "\alpha"'] & Fa \arrow[d, "\alpha"] \arrow[dl, phantom, "="] \\ & b \end{tikzcd}$$

gives

Ⓐ
$$\varphi_{a,b}(\alpha) = U\alpha \circ \varphi_{a,Fa}(1) = U\alpha \circ \eta_a .$$

Theorem 1 There is a bijection between adjunctions $(F \dashv U, \varphi)$ & natural transformations $\eta : 1 \Rightarrow UF$ whose components $\eta_a : a \longrightarrow UFa$ have the univ. prop.

▲ : given $a \xrightarrow{\alpha} Ub$ there exists a unique $\bar{\alpha} : Fa \to b$ such that

$$\begin{tikzcd}
 & UFa \arrow[dr, "U\tilde{\alpha}"] & \\
a \arrow[ur, "\eta_a"] \arrow[rr, "\alpha"'] & & Ub
\end{tikzcd}$$ commutes.

$\boxed{\text{Proof}}$ By equation Ⓐ above, given an adjunction $(F \dashv U, \varphi)$ then

$$B(Fa,b) \xrightarrow{\varphi_{a,b}} A(a,Ub)$$
$$\alpha \longmapsto U\alpha \circ \eta_A$$

& then ▲ just says that these components are bijections (which is part of def^n of adjunction).

- Furthermore, we have already seen in Ⓐ that φ is determined by η, so it suffices to show that each such η as in statement of the theorem arises from an adjunction.

Given $\eta : 1 \Rightarrow UF$ satisfying ▲ we must show that the functions

$$B(Fa,b) \xrightarrow{\varphi_{a,b}} A(a,Ub)$$
$$Fa \xrightarrow{\alpha} b \longmapsto a \xrightarrow{\eta_a} UFa \xrightarrow{U\alpha} Ub$$

give an adjunction.

They are bijections by ▲ so it remains to check naturality, but this follows from naturality of η. □

In fact, even less is required.

Theorem 2) An adjunction is specified by a functor $U : \mathcal{B} \longrightarrow \mathcal{A}$ and for each $a \in \mathcal{A}$ an object $Fa \in \mathcal{B}$ and morphism $\eta_a : a \longrightarrow UFa$ satisfying the u.p. ▲.

Proof) certainly an adjunction gives rise to this. Conversely, given Th 1, it suffices to show that the assignments $a \longmapsto Fa$ extend uniquely to a functor such that the $\eta_A : a \hookrightarrow UFa$ are components of a natural transformation.

- well, if η is to be natural, then at $\alpha : a \to b$ we must have

$$\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] \arrow[dr, dashed, bend right=80, "\eta_b \circ \alpha"'] & UFa \arrow[d, "UF\alpha"] \\
b \arrow[r, "\eta_b"'] & UFb
\end{tikzcd}$$

, but by ▲ for $a \xrightarrow{\eta_a} UFa$ at $\eta_b \circ \alpha$, there exists a unique $F\alpha$ (namely $\overline{\eta_b \circ \alpha}$) making the square commute.

- Then η will be a nat. transformation if F is a functor, which is what we must show!

- To see F pres identities consider

$$\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[d, "1_a"'] \arrow[dr, phantom, "="] & UFa \arrow[d, "U1 = 1_{UFa}"] \\
a \arrow[r, "\eta_a"'] & UFa
\end{tikzcd}$$

so $F1_a = 1_{Fa}$ by uniqueness of ▲.

- To see F pres. composition consider

$$\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] \arrow[dd, bend right=60, "\beta\circ\alpha"'] & UFa \arrow[d, "UF\alpha"] \arrow[dd, bend left=60, "U(F\beta \circ F\alpha)"] \\
b \arrow[r, "\eta_b"] \arrow[d, "\beta"'] \arrow[dr, phantom, "="] & UFb \arrow[d, "UF\beta"] \\
c \arrow[r, "\eta_c"'] & UFc
\end{tikzcd}$$

so by definition of $F(\beta\circ\alpha)$, we have $F(\beta\circ\alpha) = F\beta \circ F\alpha$. □

Corollary

$\mathcal{U} : \mathcal{B} \longrightarrow \mathcal{A}$ has a left adjoint $\iff$ $\forall a \in \mathcal{A}$ $\exists Fa \in \mathcal{B}$ and a map $\eta_a : a \longrightarrow \mathcal{U}Fa$ satisfying the univ. prop. ▲ .

Remark: This is often how people describe adjoints in practice.

Theorem) Let $\mathcal{U} : \mathcal{B} \longrightarrow \mathcal{A}$. Then its left adjoint, if it exists, is unique up to natural isomorphism.

Proof Suppose F_1, F_2 are l. adjoints to $\mathcal{U}$, with units $a \xrightarrow{\eta_{1a}} \mathcal{U}F_1a$ & $a \xrightarrow{\eta_{2a}} \mathcal{U}F_2a$ satisfying ▲ .

Then by the u.p. of $\eta_1 a$, $\exists!\, k_a : F_1a \to F_2a$ such that

$$\begin{tikzcd} & \mathcal{U}F_1a \arrow[d, "\mathcal{U}k_a"] \\ a \arrow[ur, "\eta_{1a}"] \arrow[r, "\eta_{2a}"'] & \mathcal{U}F_2a \end{tikzcd}$$

commutes. And likewise $\exists!\, l_a : F_2a \to F_1a$ s.t.

$$\begin{tikzcd} & \mathcal{U}F_2a \arrow[d, "\mathcal{U}l_a"] \\ a \arrow[ur, "\eta_{2a}"] \arrow[r, "\eta_{1a}"'] & \mathcal{U}F_1a \end{tikzcd}$$

Applying u.p. ▲ twice more, can show l_a is inverse of k_a - ie. k_a is an isomorphism.

- Naturality is an exercise for reader. □

Remark: Incredible! Forgetful Functors (v. simple) uniquely determine (up to iso) Free Functors, which describe something more complex.

Next week, limit preservation & then move on to universal algebra.