- In an adjunction $F \dashv U$, we have
$$\mathcal{A}(Fx, y) \xrightarrow{\varphi} \mathcal{B}(x, Uy)$$
natural in $x, y$.

- At $x = Uy$, obtain
$$\mathcal{A}(FUy, y) \xrightarrow{\varphi} \mathcal{B}(Uy, Uy)$$
$$FUy \xrightarrow{\varepsilon_y} y \quad \xleftarrow{\varphi^{-1}} \quad Uy \xrightarrow{1} Uy$$
called the counit.

- You can show that in general
$$\mathcal{B}(x, Uy) \xrightarrow{\varphi^{-1}} \mathcal{A}(Fx, y)$$
$$x \xrightarrow{\alpha} Uy \longmapsto Fx \xrightarrow{F\alpha} FUy \xrightarrow{\varepsilon_y} y$$

so to say that $\varphi^{-1}$ a bij$^n$ is to say:

given $\alpha : Fx \to y$ $\exists!$ $x \xrightarrow{\bar{\alpha}} Uy$ such that

$$\begin{tikzcd} & FUy \arrow[d, "\varepsilon_y"] \\ Fx \arrow[ur, "F\bar{\alpha}"] \arrow[r, phantom, "="] \arrow[dr, "\alpha"'] & \phantom{y} \\ & y \end{tikzcd}$$
.

In general, a functor $F : A \to \mathcal{B}$ has a right adj. iff for each $y \in \mathcal{B}$ $\exists$ object $Uy$ and map $FUy \xrightarrow{\varepsilon_y} y$ with the above universal property.

- In particular, to show

$U: \mathrm{Grph} \longrightarrow \mathrm{Set}$ has a right adjoint $R$, we can give give $URY \xrightarrow{\varepsilon_Y} Y$ with the prop that given

$UX \xrightarrow{\alpha} Y \quad \exists ! \, X \xrightarrow{\bar{\alpha}} RY$

such that

$$\begin{tikzcd} & URY \arrow[d, "\varepsilon_Y"] \\ UX \arrow[ur, "U\bar{\alpha}"] \arrow[r, "\alpha"] & Y \end{tikzcd}.$$

(In this case, $\varepsilon_Y = \mathrm{id}$!!)

Or, alternatively, give a map

$Y \xrightarrow{\eta_Y} URY$ sat. the univ. prop. described last week (once we have made $R$ into a functor.)