Exercise - Week 6

① Show that the forgetful functor $U: \mathrm{Mon} \to \mathrm{Set}$ does not have a right adjoint. What about groups / rings?

② The free functor $F: \mathrm{Set} \to \mathrm{Mon}$ sending a set to the list monoid is left adjoint to $U: \mathrm{Mon} \to \mathrm{Set}$. But does F have a left adjoint?

③ Consider the forgetful functor $U: \mathrm{Grph} \to \mathrm{Set}$ from graphs to sets.

need Graph to mean graph with loops, if we want Π!

Show that there are adjoint functors $\Pi \dashv F \dashv U \dashv R$ and that this string of adjunctions cannot be extended any further.

④ Consider adjoint functors

$$\mathcal{A} \underset{U_1}{\overset{F_1}{\rightleftarrows}} \mathcal{B} \underset{U_2}{\overset{F_2}{\rightleftarrows}} \mathcal{C}.$$

Show that we have an adjunction $F_1F_2 \dashv U_2U_1$.

Limits as adjoints

- Let J be a small cat & $\mathcal{C}$ a category.

- Given $a \in \mathcal{C}$ we can define the constant functor at a $\Delta_a : J \to \mathcal{C}$: $j \mapsto a$, $j \xrightarrow{g} k \mapsto a \xrightarrow{1} a$

- Show that a natural transformation $\Delta_a \xrightarrow{f} D$ is the same thing as a cone $(A \xrightarrow{f_i} D_i)_{i \in J}$.

- Show that Δ defines a functor $\mathcal{C} \xrightarrow{\Delta} [J, \mathcal{C}]$ ~ functor cat

& that Δ has a left adj. $\Leftrightarrow$ $\mathcal{C}$ has all colimits of shape J.