Lecture 6

What do adjoint functors preserve?

- Let $D : J \to \mathcal{A}$ be a diagram.
- A cone (for D) consists of an ob. L and maps $p_i : L \to D_i$ for $i \in J$ sat.

$$\begin{tikzcd} & D_i \arrow[dd, "D\alpha"] \\ L \arrow[ur, "p_i"] \arrow[dr, "p_j"'] \arrow[r, phantom, "="] & \\ & D_j \end{tikzcd} \quad \text{for} \quad \begin{tikzcd} i \arrow[d, "\alpha"] \\ j \end{tikzcd}$$

- It is a limit cone (or just that L is the limit) if it has universal property: given a cone $(k_i : A \to D_i)_{i \in I}$ $\exists !\, k : A \to L$ such that $p_i \circ k = k_i$ for all i.

- If $U : \mathcal{A} \to \mathcal{B}$ is a functor it takes the cone $(p_i : L \to D_i)_{i \in I}$ to a cone $(Up_i : UL \to UD_i)_{i \in I}$ for UD.

Def) We say that U preserves the limit L of D if the cone $(UL \xrightarrow{Up_i} UD_i)_{i \in I}$

is a limit cone.

- Similarly, we can speak of a functor preserving colimits.

Theorem

- Consider $A \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} B$.
- Then the right adjoint U preserves any limits that exist in A; the left adjoint F preserves any colimits that exist.

Proof

We have bijections $A(Fx, y) \xrightarrow{\varphi} B(x, Uy)$ natural in x & y, & $(L \xrightarrow{p_i} Di)_{i \in J}$ a limit cone for $D : J \to A$, and must show $(UL \xrightarrow{Up_i} UD_i)_{i \in J}$ is a limit cone.

- Consider cone $(X \xrightarrow{k_i} UDi)_{i \in J}$.
- Using bijections $A(Fx, Di) \xrightarrow{\varphi} B(x, UDi)$ we obtain maps
  $Fx \xrightarrow{\varphi^{-1}k_i} Di$ & claim these form a cone to D:

we must show

$$\begin{tikzcd} & Di \arrow[d, "D\alpha"] \\ FX \arrow[ur, "\varphi^{-1}k_i"] \arrow[dr, "\varphi^{-1}k_j"'] \arrow[r, phantom, "="] & {} \\ & Dj \end{tikzcd}$$

but this is equivalent to showing images of these maps $Fx \rightrightarrows Dj$ under $A(FX, Dj) \xrightarrow{\varphi} B(X, UDj)$ are equal.

- Well $\varphi\varphi^{-1}k_j = k_j$.
- $\varphi(D\alpha \circ \varphi^{-1}k_i) = UD\alpha \circ \varphi\varphi^{-1}k_i = UD\alpha \circ k_i$

by naturality of φ

so their images are the two paths of

$$\begin{tikzcd} & UDi \arrow[d, "UD\alpha"] \\ X \arrow[ur, "k_i"] \arrow[dr, "k_j"'] \arrow[r, phantom, "="] & {} \\ & UDj \end{tikzcd}$$

which agree, since the k_i are a cone.

- Since the maps $\varphi^{-1}k_i : FX \longrightarrow Di$ form a cone we obtain a unique $l : FX \longrightarrow L$ such that

$$\begin{tikzcd} FX \arrow[r, "l"] \arrow[dr, "\varphi^{-1}k_i"'] & L \arrow[d, "p_i"] \\ & Di \end{tikzcd} \quad (*)$$

For all i.

- Using the bijection $A(FX, L) \xrightarrow{\varphi} B(X, UL)$ this corresponds to a map

$X \xrightarrow{\varphi_\ell} UL$ & the equations

(*) corresp. to the equations

$$\text{(**)}\quad \begin{tikzcd} X \arrow[r, "\varphi_\ell"] & UL \arrow[d, "Up_i"] \\ K_i \arrow[r] & UD_i \end{tikzcd}$$

using naturality of φ.

- In partic, $\varphi_\ell : X \to UL$ is the unique map st. (**) commutes; therefore $(UL \xrightarrow{Up_i} UD_i)_{i \in J}$ is a limit cone. $\square$

- For example, $U : \mathrm{Grp} \to \mathrm{Set}$ preserves products, equalisers, terminal object etc. More generally, forgetful functors from algebraic cats to Set preserve all limits.

Exercise

Prove that forgetful functor $U : \mathrm{Grp} \longrightarrow \mathrm{Set}$ does not have a right adjoint.

(ie. is not a left adjoint.)

Example

- Let Field = cat. of fields & homomorphisms: preserve addition, mult., 0 & 1.
- Fields are commutative rings sat $0 \neq 1$ &
  $x \neq 0 \Rightarrow \exists \; : xy = 1$
- Because of these negative equations, Fields are not part of univ algebra
- The cat. of Fields is "bad".
  I will show the forgetful functor $U : \text{Field} \longrightarrow \text{Set}$ does not have a left adjoint.
- If it did have left adj F, then F would send the init. ob $\emptyset \in \text{Set}$ to an

init object in Field (as left adjoints preserves colimits). So it suffices to show Field does not have an initial object.

- Firstly, let $f : R \to S$ be a field homomorphism. We claim f is injective. Indeed, suppose $fx = 0$ for $x \neq 0$. Then $x \in \ker f \hookrightarrow R$ is an ideal of R, non-zero, so as R is a field, $\ker f = R$. Therefore $f1 = 0$ so $1 = f1 = 0$ which is a contradiction; hence f is injective.

- Let $\mathbb{Z}_p$ Field of integers modulo p, so $p.1 = 0$, for p a prime.
- If F is initial, $\exists$

$$\begin{tikzcd} & \mathbb{Z}_p \\ F \arrow[ur, "inj"] \arrow[dr, "inj"'] & \\ & \mathbb{Z}_q \end{tikzcd}$$

for p, q coprime.

- Since $F \hookrightarrow \mathbb{Z}_p, \mathbb{Z}_q$ are injective they reflect equations $p.1 = 0$ & $q.1 = 0$, so these equations hold in F. But as p, q coprime $1 = np + mq$, by

Bézout's identity,
so in F,

$$1 = 1.1 = np + mq.1$$
$$= n(p1) + m(q1) =$$
$$n0 + m0 = 0$$

so $1 = 0$ in F,
contradicting that F is a field. $\square$

Note: in universal algebra
all forgetful functors
have left adjoints. :)
No fields in univ. alg.

- We have seen:
  right adjoints preserve limits

Proposition

Right adjoints preserve monos.

Proof: Let $U : A \longrightarrow B$ have left adj. F, and consider mono $a \xrightarrow{f} b \in A$.

Consider $x \underset{v}{\overset{u}{\rightrightarrows}} Ua \xrightarrow{Uf} Ub$

satisfying $Uf.u = Uf.v$. We must show $u = v$.

Then we obtain maps

$$Fx \underset{\varphi^{-1}v}{\overset{\varphi^{-1}u}{\rightrightarrows}} a \xrightarrow{f} b \quad \&$$

the diagram commutes by naturality of φ.

Since f is mono, therefore
$\varphi^{-1}u = \varphi^{-1}v$.
Therefore $u=v$ so that
Uf is mono, as claimed.

Remark

If we have a right adjoint U, we know about maps from object X into UY. (i.e. corr. to maps $FX \to Y$)
Correspondingly, right adjoints U generally properties & structures defined using maps into an object:
eg. limits, monos.