Lecture 6 Part 2 - Universal Algebra

- Universal algebra : study of "sets with operations satisf. equations"

but not things like Fields

$0 \neq 1$ is not an equation!

e.g. groups, monoids, rings...

operations : $x+y$, 0, ...

equations : $x+0=x$ ...

- "Universal algebra" also called "general algebra" ...
- Two excellent books I like :
  - Algebra Chapter 0 by Aluffi
  - An invitation to General Alg. by Bergman (I am using this book a little)

Def^n) A signature $\Omega = (\Omega_n)_{n\in\mathbb{N}}$ consists of a set Ω_n for each $n \in \mathbb{N}$.

Remark) The elements of Ω_n are thought of as "n-ary operation symbol".

0-ary operations = "constants".

Examples) ① Signature Ω for monoids, groups ... has

$$\Omega = \{e, m\}.$$

i.e. $\Omega(n) = \emptyset$ for $n \neq 0, 2$.

e 0-ary op, $e \in \Omega(0)$

m binary, so $m \in \Omega(2)$

(2) Signature for rings

$\Omega = \{ \underbrace{0, 1}_{0\text{-ary}}, \underbrace{+, \times}_{2\text{-ary}} \}$

i.e. $\Omega(0) = \{0, 1\}$
$\Omega(2) = \{+, \times\}$
$\Omega(n) = \phi$ otherwise.

Def^n Let Ω be a signature.
An Ω-algebra is a set X together with, for each $n \in \mathbb{N}$ & $m \in \Omega(n)$, a function $m_X : X^n \longrightarrow X$:
$(a_1, \dots, a_n) \longmapsto m_X(a_1, \dots, a_n)$.

Remark What is X^0? Well $\underbrace{X^n = Set(n, X)}_{\text{function from the n elt set} \to X}$.

- In partic., $X^0 = Set(\underbrace{0}_{\text{empty set which is initial}}, X) \cong 1$.

- So given $e \in \Omega(0)$ & X an Ω-algebra, we have $e_X : 1 \longrightarrow X$, so this is just an element e_X of X.
- This is why elts of $\Omega(0)$ are thought of as nullary op. symbols = constants.

Notation) For an Ω-alg.

$$(X, (m_X)_{m \in \Omega n, n \in \mathbb{N}}).$$

I will denote this by $\overline{X}$, or perhaps just X.

Example

For $\Omega = \{e, m\}$
nullary, binary

an Ω-algebra is a magma:

- we have:
  - a binary function $m_X : X^2 \to X : (a,b) \mapsto m_X(a,b)$
    $e_X : 1 \to X : - \mapsto e_X \in X$

Magmas do not need to satisfy any equations.

Exercise) Find a signature Ω s.t. $\mathbb{R}$-vector spaces are certain Ω-algebras.

Def^n) A homomorphism $f : \bar{X} \to \bar{Y}$ of Ω-algebras is a function $f : X \to Y$ such that $\forall n \in \mathbb{N}$, $m \in \Omega_n$ the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{f^n} & Y^n \\ {\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle m_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes: i.e.

$f(m_X(a_1, \dots, a_n)) = m_Y(fa_1, \dots, fa_n)$ $\forall a_1, \dots, a_n \in X$

Remark This captures the general notion of homomorphism in monoids, groups, rings etc when we choose the appropriate signature.

Example If $\Omega = \{\cdot, e\}$

Example

2-ary 0-ary

A homomorphism of Ω-algs is a fn $f: X \to Y$ satisfying
$$f(a \cdot_X b) = f(a) \cdot_Y f(b) \quad \& \quad f(e_X) = e_Y.$$
(Here I write $a \cdot_X b = \cdot_X(a,b)$ for convenience.)

- There is a category Ω-Alg of Ω-algebras & Ω-algebra homomorphisms, & it has a forgetful functor
$$U: \Omega\text{-Alg} \longrightarrow \mathbf{Set}.$$
- Our goal now is to describe its left adjoint.

Term algebras

- Let Ω be a signature & X a set. We define a set $T_\Omega(X)$ of "Ω-terms" in X as follows:
- If $x \in X$ then $x \in T_\Omega(X)$;
- If $m \in \Omega_n$, $t_1, \ldots, t_n \in T_\Omega(X)$ then

the expression $\underline{m(t_1,\dots,t_n) \in T_\Omega(X)}$.

Example) $\Omega = \{ \underset{\text{2-ary}}{m}, \underset{\text{0-ary}}{e} \}$,

$X = \{a, b, c\}$

$T_\Omega(X) = \{ e, a, b, c, m(a,b), m(a,a),$
$m(b,b), m(b,a), m(a,c), \dots$
$m(a,e), \dots,$
$m(a, m(b,c)), \dots,$
$m(m(a, m(b,e)), c)$ etc... $\}$

- Remark) $T_\Omega(X)$ consists of all expression one can build from "variables" X and operation symbols in Ω.

- In fact, $\underline{T_\Omega(X) \text{ is an } \Omega\text{-algebra}}$:

given $m \in \Omega_n$, we define $T_\Omega(X)^n \xrightarrow[m]{} T_\Omega(X)$:

$(t_1,\dots,t_n) \longmapsto m(t_1,\dots,t_n)$

- We also have a function

$\eta_X : X \longrightarrow T_\Omega(X) : x \longmapsto x$.

$\boxed{\text{Theorem}}$ $U : \Omega\text{-Alg} \longrightarrow \mathrm{Set}$ has a left adjoint, whose value at X is the Ω-algebra $T_\Omega(X)$ of Ω-terms.

Proof - We have $X \to U(\underbrace{T_\Omega(X)}_{\Omega\text{-algebra}})$

- Given $X \xrightarrow{f} UY$ we must show that

$\exists !$ Ω-algebra map

$$T_\Omega(X) \xrightarrow{\bar{F}} Y \text{ such that}$$

- $$\begin{tikzcd} & U(T_\Omega(X)) \arrow[d, "U\bar{F}"] \\ X \arrow[ur, "\eta_X"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "="] & UY \end{tikzcd}$$ commutes.

This gives bij^n $\Omega\text{-Alg}(T_\Omega(X), Y) \to \mathrm{Set}(X, UY)$

- Since the triangle must commute, we must define

$$\bar{F}x = fx \text{ for } x \in X.$$

In order that $\bar{F}$ is a homomorphism we require that if $m \in \Omega_n$ then

$$\bar{F}(m(\underbrace{t_1, \dots, t_n}_{\Omega\text{-terms}})) = m_Y(\bar{F}t_1, \dots, \bar{F}t_n)$$

but this defines $\bar{F}$ on $T_\Omega(X)$ since all terms are obtained in this way inductively. $\square$

Example If $\Omega = \{m, e\}$,

$X = \{a, b, c\}$ &

$X \xrightarrow{f} UY$, we obtain

$$\begin{array}{rcl}
T_\Omega X & \xrightarrow{\bar{f}} & Y \\
a, b, c & \longmapsto & fa, fb, fc \\
e & \longmapsto & e_Y \\
m(a,b) & \longmapsto & m_Y(fa, fb) \\
m(a, m(b,c)) & \longmapsto & m_Y(fa, m_Y(fb, fc)) \\
 & & \cdots
\end{array}$$