# Exercises 8

- Let $A$ be an $\Omega$-algebra. A congruence on $A$ is an equivalence relation $E$ on $A$ such that:

(*) if $s \in \Omega_n$ & $x_1 E y_1, \dots, x_n E y_n$ then
$$s(x_1, \dots, x_n) \, E \, s(y_1, \dots, y_n).$$

(1) Explain what the condition (*) means in elementary terms if
- $\Omega = \{e, \cdot\}$ is signature for monoids
- $\Omega = \{e, \cdot, (-)^{-1}\}$ is sig. for groups.

(2) For a group $G$, show that if
- $E$ is a congruence on $G$, then the set $N_E = \{x : x E e\}$ is a normal subgroup of $G$. Show that this describes a bijection

$$\begin{array}{ccc} \mathrm{Cong}(G) & \longrightarrow & \mathrm{NormalSubgroups}(G). \\ E & \longmapsto & N_E \end{array}$$

b) Show, moreover, that
$$\underbrace{G/E}_{\text{quotient by cong.}} = \underbrace{G/N}_{\text{quotient by normal subgroup}}$$

(3) For a ring $R$, show that congruences on $R$ $\sim$ ideals on $R$

(4) Let $E$ be a congruence on $A$. Show that $E = \mathrm{Ker}(A \xrightarrow{P} A/E)$ where $A/E$ is the quotient of $A$ by $E$.