Lecture 9 - Colimits & free structures

- Last week: congruences on Ω-algebra A
- We sometimes write $\mathrm{Cong}(A)$ for set of congruences on A.
- Last week: quotients by congruences.
  This time: general quotients (aka coequalisers)

Lemma) If $(E_i)_{i\in I}$ is a set of congruences on A, then $\bigcap_{i\in I} E_i$ is also a congruence.

Proof) Easy exercise.

Proposition) Let $X \subseteq A\times A$. Then $\exists$ smallest congruence E_X containing X.

Proof) - Let $\mathcal{I} = \{E \text{ a congruence on } A : X \subseteq E\}$.
Note $\mathcal{I}$ is non-empty, as it contains $A\times A$.
- Therefore, $E_X = \bigcap_{E\in\mathcal{I}} E$ is non-empty intersection, and so a congruence. As $X \subseteq E$ for each $E\in\mathcal{I}$, $X \subseteq E_X$, as required.
- Also if $X \subseteq E$ some cong. E, then $E_X \subseteq E$ by construction. $\square$

- We call E_X the congruence generated by X.

Proposition

The category Ω-Alg has coequalisers.

Proof

- Consider $X \underset{g}{\overset{f}{\rightrightarrows}} Y \in \Omega\text{-Alg}$ &

let $E_{f,g} \subseteq Y\times Y$ be the congruence on Y

generated by $\{(fx, gx) : x \in X\}$.

- Then we have a comm. diagram

$$\begin{tikzcd} (fx, gx) & E_{f,g} \arrow[r, shift left=3, "d"] \arrow[r, shift left=1, "c"'] & Y \arrow[r, "p"] & Y/E_{f,g} \\ x \arrow[u, mapsto] & X \arrow[u] \arrow[ur, shift left=1, "f"] \arrow[ur, shift right=1, "g"'] & & \end{tikzcd}$$

where $d(x,y) = x$ & $c(x,y) = y$.

- From last week, the top row exhibits the Ω-alg $Y/E_{f,g}$ as a coequaliser of d & c.
- Must show that bottom row is coequaliser too – i.e. $pf = pg$ is a coequaliser.
- So consider $h : Y \to Z \in \Omega\text{-Alg}$ sat. $hf = hg$.
- Consider the congr. $\operatorname{Ker} h = \{(a,b) : ha = hb\}$, a congr. on Y
- Since $hfx = hgx$ for all $x \in X$, it follows that $\{(fx, gx) : x \in X\} \subseteq \operatorname{Ker} h$.
- Therefore, $E_{f,g} \subseteq \operatorname{Ker} h$: in other words, given $(x,y) \in E_{f,g}$, we have that $hx = hy$, which is to say that $hd(x,y) = hx = hy = hc(x,y)$ so $hd = hc$.

Therefore as top row is a coeq.,
we obtain a ! map

$$Y/E_{f,g} \xrightarrow{\bar{h}} Z \text{ such that}$$

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[dr, "h"'] & Y/E_{f,g} \arrow[d, "\bar{h}"] \\ & Z \end{tikzcd}$$

Therefore $Y/E_{f,g}$ is coequaliser of d & c. $\square$

Example

- coequalisers capture "presentations".

Often one speaks of an algebra

$$\langle x_1, \dots, x_n \mid s_1 = t_1, \dots, s_m = t_m \rangle$$

generated by n elements $x_1, \dots, x_n$ subject to equations $s_i, t_i \in T_\Sigma(x_1, \dots, x_n)$.

- Its <u>universal defining property</u> is that maps

$$\langle x_1, \dots, x_n \mid s_1 = t_1, \dots, s_m = t_m \rangle \longrightarrow Y$$

correspond to n-tuples $\{x_1, \dots, x_n\} \xrightarrow{v} Y$ in Y

satisfying the equations above.

- The terms (s_i, t_i) specify functions

$$\begin{tikzcd} m \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & T_\Omega n = U_\Omega F_\Omega n \end{tikzcd}$$

m-elt set

$$i \longmapsto s_i, t_i,$$ respectively.

By adjointness, these correspond to functions

$$\begin{tikzcd} F_\Omega m \arrow[r, shift left, "\bar{s}"] \arrow[r, shift right, "\bar{t}"'] & F_\Omega n \in \Omega\text{-Alg} \end{tikzcd}$$

whose coequaliser is the algebra

$$\langle x_1, \ldots, x_n \mid s_1 = t_1, \ldots, s_m = t_m \rangle.$$

Proposition

The category Ω-Alg has all small coproducts.

Proof) (Sketch).

- Consider family $(X_i)_{i \in I}$ of Ω-algebras.
- We need to find an ob $C \in \Omega$-Alg & maps $(X_i \xrightarrow{\iota_i} C)_{i \in I}$ such that:
- given $(X_i \xrightarrow{f_i} A)_{i \in I}$ $\exists ! \, f : C \longrightarrow A$ such that $f \circ \iota_i = f_i \; \forall i \in I$.

Form coproduct (disjoint union) $X = \bigcup_{i \in I} X_i$ in Set, & then $F_\Omega(X)$.

Then we have

$$\begin{tikzcd} X_i \arrow[r, "p_i"] \arrow[rr, bend right, "k_i"'] & \bigcup_{i\in I} X_i = X \arrow[r, "\eta_X"] & F_\Omega(X) \end{tikzcd}$$ but

problem is that k_i need not be an Ω-alg homomorphism.

- To fix this, consider congruence E on $F_\Omega X$ generated by

$$\{ (k_i(s(x_1, \dots, x_n)), s(k_i x_1, \dots, k_i x_n)) : n \in \mathbb{N}, i \in I, s \in \Omega_n, x_1, \dots, x_n \in X_i \}$$

& then each composite

$$\begin{tikzcd} X_i \arrow[r, "k_i"] \arrow[rr, bend right, "l_i"'] & F_\Omega X \arrow[r, "p"] & F_\Omega X / E = C \end{tikzcd}$$

has each l_i a homomorphism.

Then, straightforward to check C is coproduct. $\square$

- Since all colimits can be constructed from coproducts and coequalisers, we have:

Corollary) Ω-Alg has all colimits.

- This is dual to result of last week that R-Alg has limits.

(Ω, E)-Algebras & their good properties

Proposition

For Ω a signature & E a set of equations, the full subcategory $(\Omega,E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ is closed under products, subalgebras and quotients (aka homomorphic images.)

Proof

- Let $(s,t) \in E$ be an equation in variables X. These $s,t \in F_\Omega X$.
- Then an Ω-alg $A \models s = t \iff$ each homomorphism of Ω-algebras $f : F_\Omega X \to A$ satisfies $f(s) = f(t)$.
- Let $A \in (\Omega,E)\text{-Alg}$ & $B \overset{i}{\hookrightarrow} A$ a subalgebra. Consider $f : F_\Omega X \to B$. Then as A is (Ω,E)-alg, $if(s) = if(t)$. As i injective, $f(s) = f(t)$, so B is (Ω,E)-algebra.
- Consider product $\prod_{i \in I} A_i$ of (Ω,E)-algebras & $f : F_\Omega X \to \prod_{i\in I} A_i$. Must show $fs = ft$. Then as $p_i : \prod_{i \in I} A_i \to A_i$ is a homomorphism, $p_i \circ f = f_i$ is a homomorphism so $(fs)_i = f_i s \underset{\substack{\text{since } A_i \\ (\Omega,E)\text{-alg}}}{=} f_i t = (ft)_i$ for all i. Hence $\prod A_i \models s = t$.
- Let $A \overset{p}{\twoheadrightarrow} B \in \Omega\text{-Alg}$ be surjective & A an (Ω,E)-alg. Consider $f : F_\Omega X \to B$. Must show $fs = ft$.

In fact, $\exists$ a factorisation

$$\begin{tikzcd} & A \arrow[d, "p"] \\ F_\Omega X \arrow[ur, "\bar{f}"] \arrow[r, "f"'] \arrow[ur, phantom, "="] & B \end{tikzcd}$$

defined as follows: at $x \in X$, since p is surj. $\exists \bar{f}x$ such that $p(\bar{f}x) = fx$. Then by univ prop. of $F_\Omega X$, this extends to a hom. $\bar{f} : F_\Omega X \to A$ and $p \circ \bar{f} = f$ since $p \circ \bar{f} \circ \eta = f \circ \eta$ where $\eta : X \to U_\Omega F_\Omega X$ is the inclusion, using the univ.

property of $\overline{F_\Omega X}$

But then $\bar{f}s = \bar{f}t$ as A is (Ω, E)-alg, so $fs = p\bar{f}s = p\bar{f}t = ft$. Therefore B is (Ω, E)-alg too. $\square$

Technical note on quotients

- let's write $\mathrm{SurjQuot}(A)$ for class of surjective homomorphisms $A \xrightarrow{f} B$ for B any object.
- This is a proper class - indeed just consider maps to terminal objects (1 element sets).
- let us say $(f, B), (g, C) \in \mathrm{SurjQuot}(A)$ are iso. if $\exists$ iso $B \xrightarrow{h} C$ such that

$$\begin{tikzcd} & B \arrow[d, "h"] \\ A \arrow[ur, "f"] \arrow[r, "g"'] & C \end{tikzcd}$$

, & write $(f, B) \sim (g, C)$.

- By first iso. theorem (last week), $(f, B) \cong (A/\ker f, p_{\ker f})$ so we have a surj. function

$$\begin{array}{ccc} \mathrm{Cong}(A) & \longrightarrow & \mathrm{SurjQuot}(A)/\sim \\ E & \longmapsto & A \xrightarrow{p} A/E \end{array}$$

(In fact this is a bij^n)

- Since $\mathrm{Cong}(A) \subseteq \mathrm{Powerset}(A \times A)$ is a set, therefore $\mathrm{SurjQuot}(A)/\sim$ is a set : in particular, there is only a set of surjective quotients of A up to isomorphism.

We would like to prove:

Theorem) The inclusion $(\Omega,E)\text{-Alg} \xhookrightarrow{i} \Omega\text{-Alg}$ has a left adjoint.

- will prove it as a special case of a more general result, which also applies to Hausdorff $\hookrightarrow$ Top and sim. situations.

Theorem

Let $i : \mathcal{B} \hookrightarrow \mathcal{C}$ be inclusion of full subcat.

Suppose:

- $\mathcal{C}$ has two classes $\mathcal{E}$ & $\mathcal{M}$ of arrows with the properties that:
  - Each $e \in \mathcal{E}$ is epi
  - Each $f : A \to B \in \mathcal{C}$ can be written as $f = me$ where $m \in \mathcal{M}$ & $e \in \mathcal{E}$.
  - For each $A \in \mathcal{C}$, $\mathcal{E}\text{-Quot}(A)/\sim$ is a set.
- Suppose also that $\mathcal{C}$ has products & $\mathcal{B}$ is closed in $\mathcal{C}$ under products.
- If $A \xrightarrow{f} B \in \mathcal{C}$ has $f \in \mathcal{M}$ & $B \in \mathcal{B}$ then $A \in \mathcal{B}$

Then $i : \mathcal{B} \hookrightarrow \mathcal{C}$ has left adjoint R & the unit $X \xrightarrow{\eta_X} iRX \in \mathcal{E}$.

Before proving theorem, let's check it applies to $(\Omega,E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$.

- $\mathcal{E}$ = surjective homs, $\mathcal{M}$ = subalgebra inclusions
- Each surj. hom. is epi.
- Factor $f : A \to B$ through $\mathrm{im} f \hookrightarrow B$ as (surj / subalg. incl.)
- $\mathrm{SurjQuot}(A)$ is a set.
- $(\Omega,E)\text{-Alg}$ closed under prods & subalgebras

so the Theorem will apply.

Proof

- Notation: As i inclusion, write $iA = A$.
- We must show that given $X \in \mathcal{C}$ $\exists RX \in \mathcal{B}$ and $X \xrightarrow{\eta_X} RX \in \mathcal{E}$ such that: given $X \xrightarrow{g} Y$ where $Y \in \mathcal{B}$ then $\exists!\, RX \xrightarrow{\bar{g}} Y$ such that $\bar{g} \circ \eta_X = g$.
- Since $\mathcal{E}\text{-Quot}(X)/\sim$ is a set, there exists a set
$$(X \xrightarrow{f_i \in \mathcal{E}} X_i : i \in I)$$ such that each $X_i \in \mathcal{B}$ and if $X \xrightarrow{f \in \mathcal{E}} Y \in \mathcal{B}$ then $(f, Y) \sim (f_i, X_i)$ some X_i.
- Form the product
$$\begin{tikzcd} X \arrow[r, "\exists! f"] \arrow[dr, "f_i"'] & \prod_{i \in I} X_i \in \mathcal{B} \arrow[d, "p_i"] \\ & X_i \in \mathcal{B} \end{tikzcd}$$
which belongs to $\mathcal{B}$ since $\mathcal{B}$ closed under prods.
- Now factor
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\eta_X \in \mathcal{E}"'] & & \prod X_i \\ & RX \arrow[ur, "m \in \mathcal{M}"'] & \end{tikzcd}$$
Then $RX \in \mathcal{B}$ since $m \in \mathcal{M}$.

• Let $g : X \to Y \in \mathcal{C}$ with $Y \in \mathcal{B}$.
If g factors through η_X, it does so uniquely since $\eta_X \in \mathcal{E}$ is epi.
Therefore

we only need to check that there exists a factorisation of g through η_X.

- Factor $X \xrightarrow{g} Y$

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "g_1 \in \mathcal{E}"'] & & Y \\
& Z \arrow[ur, "g_2 \in \mathcal{M}"'] &
\end{tikzcd}$$

Then as $Y \in \mathcal{B}$ & $g_2 \in \mathcal{M}$, $Z \in \mathcal{B}$ too. Therefore

$$\begin{tikzcd}
\exists\ X \arrow[r, "f_i"] \arrow[dr, "g_1"'] & X_i \arrow[d, "h \text{ an iso}"] \\
& Z
\end{tikzcd}$$

But then we have

$$\begin{tikzcd}
RX \arrow[r, "m"] & \prod X_i \arrow[r, "p_i"] & X_i \arrow[d, "h"] & \\
& & Z \arrow[dr, "g_2"] & \\
X \arrow[uu, "\eta_X"] \arrow[uur, "f"'] \arrow[uurr, "f_i"] \arrow[ur, "g_1"'] \arrow[rrr, "g"'] & & & Y
\end{tikzcd}$$

so g factors through RX, as required. $\square$

Corollary 1 Inclusion

$$(\Omega, E)\text{-Alg} \xhookrightarrow{i} \Omega\text{-Alg}$$

has a left adjoint, and

each unit component

$$X \xrightarrow{\eta_X} RX$$ is surj.

homomorphism.

Corollary 2 $U : (\Omega, E)\text{-Alg} \longrightarrow \mathbf{Set}$ has a left adjoint.

Proof

$$\begin{tikzcd}
(\Omega,E)\text{-Alg} \arrow[rr, hook, "i"', shift right] \arrow[dr, "U"'] & & \Omega\text{-Alg} \arrow[ll, "R"', shift right] \arrow[ll, phantom, "\perp"] \arrow[dl, "U_\Omega", shift left] \\
& \mathbf{Set} \arrow[ur, "F_\Omega", shift left] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

$''$

& the adjoints compose:

$RF_\Omega \dashv U_\Omega i = U$, ie.

$(\Omega, E)\text{-Alg}(RF_\Omega X, Y) \cong$

$\Omega\text{-Alg}(F_\Omega X, iY) \cong$

$\mathbf{Set}(X, U_\Omega iY) =$

$\mathbf{Set}(X, UY)$.

So $RF_\Omega \dashv U$. $\square$

Next week: minor modifications

to prove Birkhoff's theorem, & then start module theory.