Exercises 9

The following questions explore some topics beyond the lectures.

Let $(t,s) \in T_\Omega X$.

These correspond to maps

1 element set

$$1 \overset{t}{\underset{s}{\rightrightarrows}} T_\Omega X = U_\Omega F_\Omega X$$ and so

to maps $F_\Omega 1 \overset{\bar{t}}{\underset{\bar{s}}{\rightrightarrows}} F_\Omega X \in \Omega\text{-Alg}$, whose coequaliser I will denote by

$$F_\Omega X \xrightarrow{p_{t,s}} F_\Omega X / t=s .$$

(1) Show that an Ω-algebra $A \models t=s$ $\iff$ Each $F_\Omega X \xrightarrow{f} A$ admits a unique extension along $p_{t,s}$ as below

$$\begin{tikzcd} F_\Omega X \arrow[r, "p_{t,s}"] \arrow[dr, "f"'] & F_\Omega X / t=s \arrow[d, dashed, "\exists ! \bar{f}"] \\ & A \end{tikzcd}$$

(One writes $p_{t,s} \perp A$ — this is the categorical notion of orthogonality.)

(2) A related notion is orthogonality of morphisms:

given arrows

$f : A \longrightarrow B$ & $g : C \longrightarrow D$

in a category $\mathcal{C}$ we say <u>f is orthogonal to g</u> ($f \perp g$)

if each square

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "s"'] & D \end{tikzcd}$$

has a unique filler t:

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "s"'] \arrow[ur, "t"] & D \end{tikzcd}$$

Show that in (Ω, E)-Alg.

<u>surjective</u> homs $\perp$ <u>injective</u> homs.

③ Given a class $\mathcal{E}$ of morphisms in $\mathcal{C}$, let

$$\mathcal{E}^{\perp} = \{ f : e \perp f \text{ for each } e \in \mathcal{E} \}$$

& ${}^{\perp}\mathcal{E} = \{ f : f \perp e \text{ each } e \in \mathcal{E} \}$.
Show that both classes are closed under composition & contain the isos.

(4) A factorisation system on $\mathcal{C}$ consists of two classes of maps $\mathcal{E}$ & $\mathcal{M}$ with

- $\mathcal{M} = \mathcal{E}^{\perp}$, $\mathcal{E} = {}^{\perp}\mathcal{M}$ &

each $A \xrightarrow{f} B$ can be factored as $f = me$ with $e \in \mathcal{E}$ & $m \in \mathcal{M}$.

a) Show that (surjective / injective) forms a factorisation system on (Ω, E)-Alg.

b) Show that the factorisation $f = me$ as $e \in \mathcal{E}$ followed by $m \in \mathcal{M}$

in a Factorisation system

is unique up to unique isomorphism.