Lecture 12

Free modules

- The forgetful functor $U : \mathrm{Mod}_R \longrightarrow \mathrm{Set}$ has a left adjoint (as for any category of the form (Ω, E)-Alg), which assigns to a set X the free R-module FX.
- FX has a simple description: elements of FX are "formal linear combinations"

  $$r_1 x_1 + \dots + r_n x_n \quad \text{where } r_i \in R, x_i \in X$$

  with obvious R-module structure.
- The unit $X \xrightarrow{\eta_X} UFX : x \longmapsto x$ & given $X \xrightarrow{f} UY$ there exists a unique R-mod. map $\bar{f} : FX \longrightarrow Y$ st $\bar{f} \circ \eta = f$ - ie. $\bar{f}(x) = f(x)$. Indeed, in order for $\bar{f}$ to be R-mod map we must define $\bar{f}(r_1 x_1 + \dots + r_n x_n) = r_1 f(x_1) + \dots + r_n (x_n)$ & it easy to see $\bar{f}$ is R-mod. homomorph.

Remark

- By formal linear combin $r_1 x_1 + \dots + r_n x_n$ one really just means

  $$((r_1, x_1), \dots, (r_n, x_n)).$$

Exercise) Check $FX \cong \bigoplus_X R$

(so in particular $F1 \cong R$).

Tensor products of R-modules

- Let R be a commutative ring, M, N, L R-modules.
- A bilinear map is a function
$$F : M \times N \longrightarrow L \quad \left(\begin{array}{l}\text{not an } R\text{-mod.}\\ \text{map (aka. linear maps)}\end{array}\right)$$
- such that for $m \in M$,
$$F(m, -) : N \longrightarrow L \text{ is linear,}$$
- & for $n \in N$,
$$F(-, n) : M \longrightarrow L \text{ is linear.}$$

(In other words, F is "linear in each variable".

- In other words,
$$\left\{\begin{array}{l} - F(m, a+b) = F(m,a) + F(m,b) \\ - F(m, ra) = r.F(m,a) \\ - F(a+b, n) = F(a,n) + F(b,n) \\ - F(ra, n) = r.F(a,n). \end{array}\right.$$

- Let $Bil(M, N; L)$ denote the set of bilinear maps $M \times N \longrightarrow L$.

Def^n) The tensor product
$$\otimes : M \times N \longrightarrow M \otimes N$$
is a universal bilinear map: that is, given $F : M \times N \longrightarrow L$ bilinear $\exists!$ linear map $M \otimes N \xrightarrow{\bar{F}} L$ such

that the triangle

$$\begin{tikzcd} M\times N \arrow[d, "\theta"'] \arrow[dr, "F"] & \\ M\otimes N \arrow[r, "\text{``}\bar{F}\text{''}"'] & L \end{tikzcd}$$

commutes.

Remarks

- This property characterise the Tensor product uniquely up to isomorphiom
- The definition equivalently says that we have a bijection
  $\mathrm{Bil}(M,N;L) \cong \mathrm{Mod}_R(M\otimes N, L)$
  natural in L
- We have not yet proved the existence of $M\otimes N$ (which we will do).
  It's main properties follow from simply it's defining universal property.
- Let's prove them now.

Theorem

Let R be a comm. ring.

(1) The tensor product extends to a functor $\mathrm{Mod}_R \times \mathrm{Mod}_R \xrightarrow{\otimes} \mathrm{Mod}_R$

(2) We have (natural) isomorphisms of R-modules
$M\otimes N \cong N\otimes M$, $R\otimes M \cong M$, $M\otimes R \cong M$.

Remark: Also have isomorphisms $(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$ which won't prove here (though not too hard!)

Proof

- Firstly, by $\mathrm{Mod}_R \times \mathrm{Mod}_R$ I mean the cartesian product of categories:
objects - pairs (A, B) of R-mods,
arrows - $(f, g) : (A, B) \longrightarrow (C, D)$ are pairs $f : A \to C$ & $g : B \to D$.
- Given (f, g), as above, we obtain fn
$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{\theta_{C,D}} C \otimes D$$
& this is bilinear:
- $\theta_{C,D} \circ (f \times g)(a, -) = \theta_{C,D}(fa, -) \circ g$ is a comp of two linear maps & so linear,
- Sim linearity in second variable.
- Therefore by the u.p. of $A \otimes B$ we obtain a unique $A \otimes B \xrightarrow{f \otimes g} C \otimes D$ such that
$$\begin{tikzcd} A \times B \arrow[r, "f \times g"] \arrow[d, "\theta_{A,B}"'] & C \times D \arrow[d, "\theta_{C,D}"] \\ A \otimes B \arrow[r, "f \otimes g"] & C \otimes D \end{tikzcd}$$
commutes.

- Uniqueness implies functoriality (Check this!)

- Next, we construct iso
$$\bar{s}_{A,B} : A \otimes B \longrightarrow B \otimes A.$$
For this, consider bilin. map
$$A \times B \xrightarrow{s_{A,B}} B \times A \xrightarrow{\otimes_{B,A}} B \otimes A$$
where $s_{A,B}(a,b) = (b,a)$.
This induces a unique lin map
$$A \otimes B \xrightarrow{\bar{s}_{A,B}} B \otimes A \text{ such that}$$

$$\begin{tikzcd}
A \times B \arrow[r, "s_{A,B}"] \arrow[d, "\otimes_{A,B}"'] & B \times A \arrow[d, "\otimes_{B,A}"] \\
A \otimes B \arrow[r, "\bar{s}_{A,B}"'] & B \otimes A
\end{tikzcd}$$
commutes.

By the universal property (ie. uniqueness) to see $\bar{s}_{B,A} \circ \bar{s}_{A,B} = id_{A \otimes B}$ so that $\bar{s}_{A,B}$ is iso.

- Next, let's show $R \otimes A \cong A$.
Consider the function
$$R \times A \xrightarrow{K} A$$
$$(r,a) \longmapsto ra$$
which is clearly bilinear (as A is R-module).

- Consider $R \times A \xrightarrow{F} B$ bilinear.

- Then each $F(-,a) : R \longrightarrow B$ is linear so $F(r,a) = F(r.1,a) = r.F(1,a)$ & we obtain a unique factorisation

$$\begin{tikzcd} R \times A \arrow[rr, "F"] \arrow[dr, "k"'] & & B \\ & A \arrow[ur, "{F(1,-)}"'] & \end{tikzcd}$$

as

$F(1,-)k(r,a)$
$= F(1,-)\, r.a$
$= F(1, r.a)$
$= r.F(1,a)$
$= F(r,a)$.

so that

$$R \times A \xrightarrow{k} A$$

is universal bilinear ;
so $\exists!$ iso $R \otimes A \xrightarrow{\ell} A$ such that

$$\begin{tikzcd} R \times A \arrow[r, "\theta"] \arrow[dr, "k"'] & R \otimes A \arrow[d, "\ell", "\wr"'] \\ & A \end{tikzcd}$$

commutes.

- Finally, we have composite iso
$A \otimes R \cong R \otimes A \cong A$. $\square$

Construction of the tensor product

- Write

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right, "F"'] \end{tikzcd}.$$

- Given R-modules M, N consider the free R-module $F(UM \times UN)$, so have a function

$$\begin{aligned} UM \times UN &\xrightarrow{\ \eta\ } UF(UM \times UN) \\ (a,b) &\longmapsto (a,b). \end{aligned}$$

- This is not bilinear as we would require:

$$\left\{ \begin{aligned} (a, b+c) &= (a,b) + (a,c) \\ (a, rb) &= r.(a,b) \\ (ra, b) &= r.(a,b) \\ (a+b, c) &= (a,b) + (a,c) \end{aligned} \right\}$$

& these do not hold in $F(UM \times UN)$.

- To force them to hold we consider submodule of $F(UM \times UN)$

$$R = \left\langle \begin{aligned} &(a, b+c) - ((a,b) + (a,c)), \\ &(a, rb) - r.(a,b), \\ &(ra, b) - r.(a,b), \\ &(a+b, c) - ((a,b) + (a,c)) \ \forall \\ &r, a, b, c. \end{aligned} \right\rangle.$$

- So $R \leqslant F(UM \times UN)$ is

a submodule gen. by these elts.
& then form
$$F(UM \times UN) \xrightarrow{p} F(UM \times UN)/R.$$
Then by construction
$$\begin{tikzcd} UM \times UN \arrow[r, "\eta"] \arrow[rr, bend right, "\otimes"'] & UF(UM \times UN) \arrow[r, "Up"] & U(F(UM \times UN)/R) \end{tikzcd}$$
is bilinear since it sends all elements of R to 0.

Theorem
$$\begin{tikzcd} UM \times UN \arrow[r, "\eta"] \arrow[rr, bend right, "\otimes"'] & UF(UM \times UN) \arrow[r, "Up"] & U(F(UM \times UN)/R) \end{tikzcd}$$
is universal bilinear map.

Proof - Given any function
$UM \times UN \xrightarrow{H} UA$ for A an R-mod.
- Then $\exists!$ lin. map $F(UM \times UN) \xrightarrow{\bar{H}} A$ such that $U\bar{H} \circ \eta = H$ (ie. $\bar{H}(a,b) = H(a,b)$.)
- Then H is bilinear $\Leftrightarrow$
$\bar{H}$ sends elts of R to 0 $\Leftrightarrow$
$\bar{H}$ factors through quotient map p.
- In partic., if H is bilinear, we obtain a unique R-module map $F(UM \times UN)/R \xrightarrow{\bar{\bar{H}}} A$ such that

$$
\begin{tikzcd}
UM \times UN \arrow[rr, "\otimes"] \arrow[drr, "H"'] & & U(F(UM \times UN)/R) \arrow[d, "U\overline{H}"] \\
& & UA
\end{tikzcd}
$$

so $\otimes$ is universal bilinear $\square$

Remark

- Explicitly, we have

$$M \otimes N = F(UM \times UN)/R$$

has generators written as

$$a \otimes b := (a, b) \text{ for } a \in M, b \in N$$

sub. to relations

$$
\begin{aligned}
a \otimes (b + c) &= a \otimes b + a \otimes c \\
a \otimes (rb) &= r(a \otimes b) \\
(a + b) \otimes c &= a \otimes c + b \otimes c \\
ra \otimes c &= r(a \otimes c).
\end{aligned}
$$

- Not every element of $M \otimes N$ is of form $a \otimes b$ — have seen

$$a_1 \otimes b_1 + a_2 \otimes b_2 \ldots$$

- In any case, the tensor product exists!

The internal hom in Mod_R

<u>Proposition</u>

Let $M, N \in \mathrm{Mod}_R$. Then the set $\mathrm{Mod}_R(M,N)$ of linear maps is an <u>R-module</u> when we define $(f+g)(a) = f(a)+g(a)$ &

$$(r.f)(a) = r.f(a).$$

<u>Proof</u>

Let's check $f+g \in \mathrm{Mod}_R$:

- $(f+g)(a+b) = f(a+b) + g(a+b)$
$= f(a)+f(b)+g(a)+g(b)$
(as N abelian) $= f(a)+g(a)+f(b)+g(b)$
$= (f+g)(a) + (f+g)(b)$.
- $(f+g)(ra) = f(ra) + g(ra)$
$= rf(a) + rg(a)$
$= r((f+g)(a))$.

So $f+g \in \mathrm{Mod}_R$.

- Sim $r.f \in \mathrm{Mod}_R$ so we have an R-module str. on $\mathrm{Mod}_R(M,N)$. $\square$

- For fixed $M \in \mathrm{Mod}_R$, we can define a <u>functor</u>

$$\mathrm{Mod}_R(M,-) : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$$

$$N \longmapsto \mathrm{Mod}_R(M,N)$$

&

$$N \xrightarrow{f} L \longmapsto \mathrm{Mod}_R(M,N) \longrightarrow \mathrm{Mod}_R(M,L)$$

$$M \xrightarrow{g} N \longmapsto M \xrightarrow{fg} L$$

& it is easy to see this is a functor
(follows from category axioms for Mod_R)

- We know $\otimes : \mathrm{Mod}_R \times \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ is a functor & we obtain functors
$M \otimes - : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R : N \longmapsto M \otimes N$
$N \xrightarrow{f} L \longmapsto M \otimes N \xrightarrow{1 \otimes f} M \otimes L$
& similarly
$- \otimes M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ sending
$$N \longmapsto N \otimes M$$
$$N \xrightarrow{f} L \longmapsto N \otimes M \xrightarrow{f \otimes 1} L \otimes M.$$

Proposition

We have adjunctions
$$- \otimes B \dashv \mathrm{Mod}_R(B, -) \quad \&$$
$$B \otimes - \dashv \mathrm{Mod}_R(B, -).$$

Proof

For $- \otimes B \dashv \mathrm{Mod}_R(B,-)$ to be an adjunction we need a bijection
$$\mathrm{Mod}_R(A \otimes B, C) \cong \mathrm{Mod}_R(A, \mathrm{Mod}_R(B, C))$$
nat in A, C.

To see this, we have a bij^n between lin maps $A \otimes B \xrightarrow{F} C$ &

bilin maps $A \times B \xrightarrow{F} C$ &

lin. maps $A \xrightarrow{\overline{\overline{F}}} \mathrm{Mod}_R(B, C)$

where

$$\overline{\overline{F}}(a) : B \longrightarrow C$$
$$b \longmapsto F(a, b).$$

Here bilin of F in A corresponds to linearity of $\overline{\overline{F}}$

& bilin of F in B corresponds to each $\overline{\overline{F}}(a) : B \longrightarrow C$ being linear.

- For $B \otimes - \dashv \mathrm{Mod}_R(B, -)$ we use composite bijection

$$\mathrm{Mod}_R(B \otimes A, C) \cong \mathrm{Mod}_R(A \otimes B, C) \cong \mathrm{Mod}_R(A, \mathrm{Mod}_R(B, C)).$$

$\square$

Corollary

Both $A \otimes -$, $- \otimes A : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ preserve colimits (in particular direct sums).

$\boxed{\text{Proof}}$ By prev. prop., they are left adjoints & left adjoints preserve colimits.

The above says

$$A \otimes \left(\bigoplus_{i \in I} B_i \right) \cong \bigoplus_{i \in I} \left(A \otimes B_i \right)$$

&

$$\left(\bigoplus_{i \in I} B_i \right) \otimes A \cong \bigoplus_{i \in I} \left(B_i \otimes A \right).$$

Corollary

$$R^m \otimes R^n \cong R^{mn}$$

Proof

$$R^m \otimes R^n \cong R^m \otimes \big(\overbrace{R \oplus \cdots \oplus R}^{n \text{ times}} \big)$$

$$\cong \underbrace{(R^m \otimes R) \oplus \cdots \oplus (R^m \otimes R)}_{n \text{ Times}} \quad \text{by the above}$$

$$\cong \underbrace{R^m \oplus \cdots \oplus R^m}_{n \text{ Times}} \quad \text{as } A \otimes R \cong A$$

$$\cong R^{mn} \quad \text{as} \quad R^m \cong \underbrace{R \oplus \cdots \oplus R}_{m \text{ Times}}.$$

$\square$

Exercise class: examples.