M7777 Applied Functional Data Analysis
12. Sparse FDA

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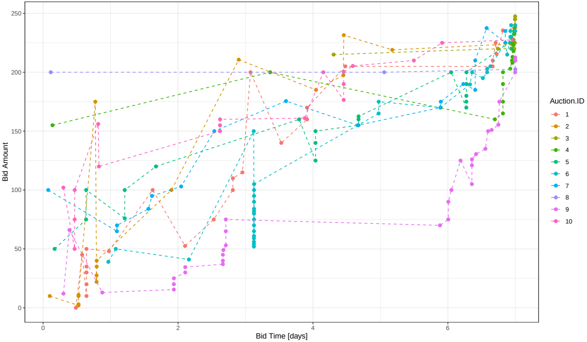
Sparse FDA

Ebay Auctions

Jank and Shmueli, 2007

- 7-Day auctions for new Palm M515 PDAs
- 149 Auctions, collected May-June 2003



Sample of 10 auctions – bid histories.

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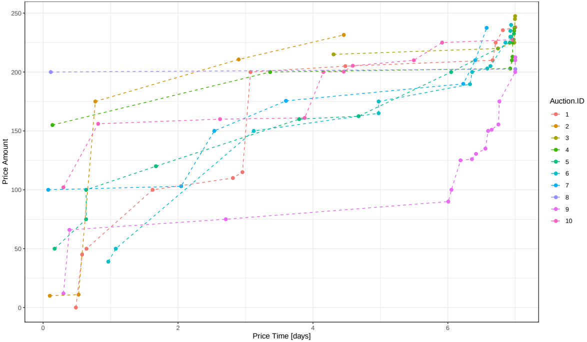
Auction Price

- Only increases if bid is greater than current price



Sample of 10 auctions – price histories.

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We will consider a model

$$Y_{ij} = \underbrace{\mu(t_{ij}) + \varepsilon_i(t_{ij})}_{X_i(t_{ij})} + \delta_{ij},$$

for $1 \leq i \leq n$, $1 \leq j \leq n_i$, with assumptions

$\mu(t)$ ... the mean function (required to be smooth)

$\varepsilon_i(t)$ ... subject specific error functions, induce correlation between observations on the same subject, let's denote $c(s,t) = \text{Cov}(X(s), X(t)) = \text{Cov}(\varepsilon(s), \varepsilon(t))$

δ_{ij} ... errors explaining measurement noise, iid across both i and j, let's denote $\text{Var}(\delta_{ij}) = \sigma^2(t_{ij})$.

It means, that we observe a process $Y(t)$ in n samples $X_i(t)$, the i-th sample is observed in times $t_1, \dots, t_{n_i}$ with setting

$$\text{Cov}(Y(s), Y(t)) = c(s,t) + \sigma^2(s) I_{s=t}.$$

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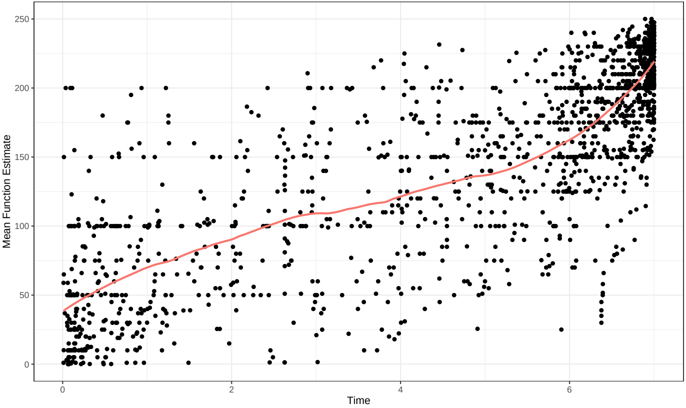
The Main Idea

1. Let us consider all measurements Y_{ij}, $1 \leq i \leq n$, $1 \leq j \leq n_i$
2. Get an estimate $\hat{\mu}(t)$ of the mean function $\mu(t)$ (nonparametric, e.g. local linear kernel smoother, spline smoothing etc.)



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3. Let us consider a set of time points pairs

$$\mathbf{T} = \{(t_{ij_1}, t_{ij_2}) : 1 \leq i \leq n, 1 \leq j_1 \leq n_i, 1 \leq j_2 \leq n_i, j_1 \neq j_2\}$$

with its values

$$Z(t_{ij_1}, t_{ij_2}) = (Y_{ij_1} - \hat{\mu}(t_{ij_1}))(Y_{ij_2} - \hat{\mu}(t_{ij_2})),\ (t_{ij_1}, t_{ij_2}) \in \mathbf{T}$$

and get the covariance surface estimate $\hat{c}(s, t)$ (bivariate local linear etc.).

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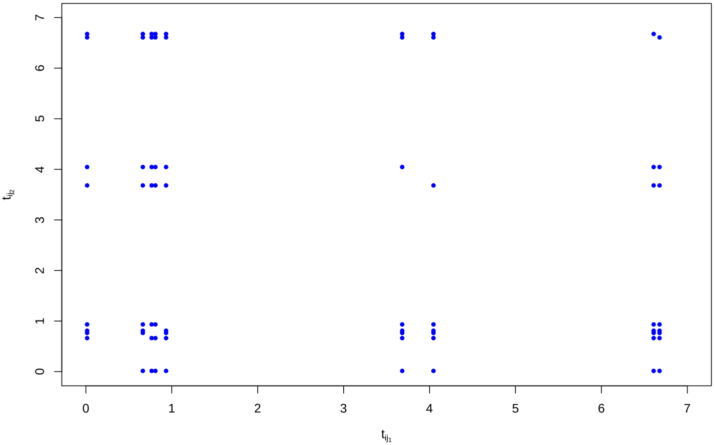
Samples: 1



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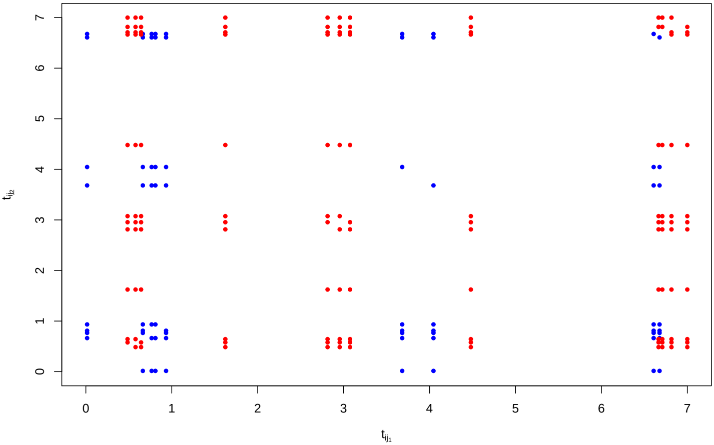
Samples: 2



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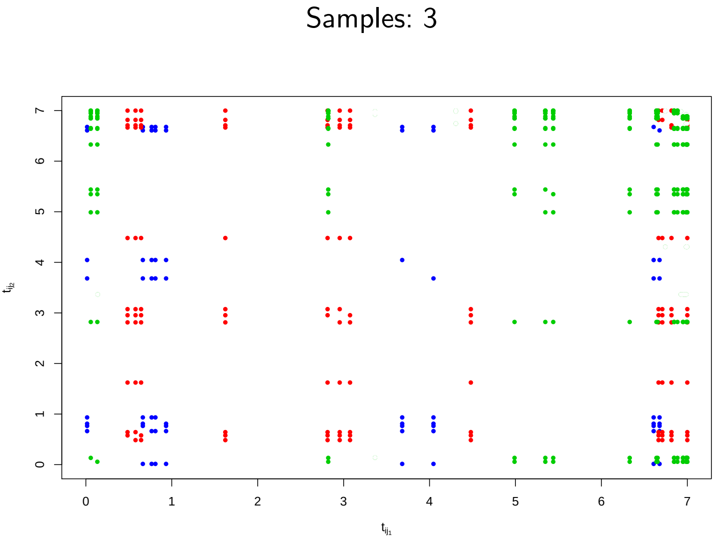
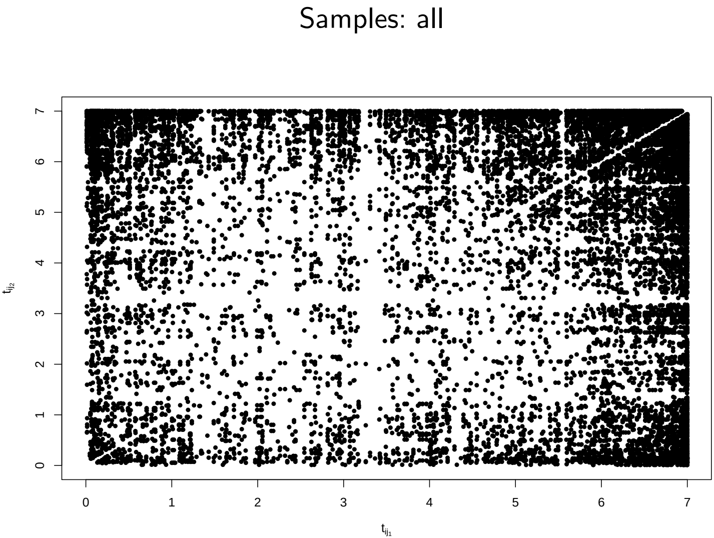


Sparse FDA



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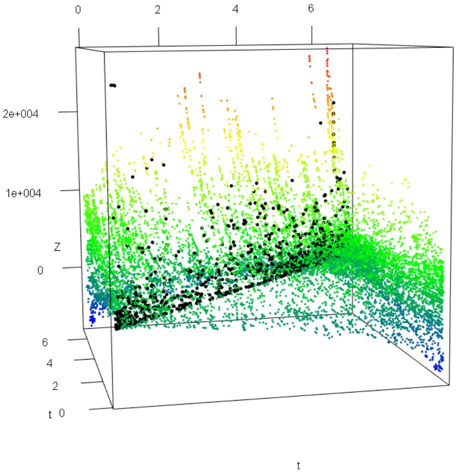
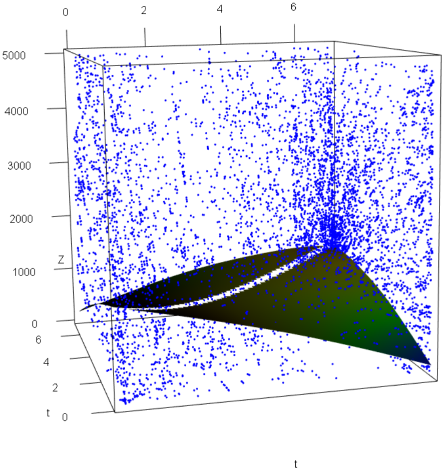
Auction Price



Raw Covariance Plot, black – diagonal terms

Auction Price



Covariance Estimate $\hat{c}(s, t)$, diagonal exluded

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4. Take diagonal terms only

$$\mathbf{T}_{diag} = \{(t_{ij}, t_{ij}) : 1 \leq i \leq n, 1 \leq j \leq n_i\} \text{ and its } Z(t_{ij}, t_{ij})$$

and by a univariate smoother get $\tilde{c}(t,t)$. Thus, an estimate of $\sigma^2(t)$

$$\hat{\sigma}^2(t) = \tilde{c}(t,t) - \hat{c}(t,t).$$

5. The estimate of $\mathrm{Cov}(Y(s), Y(t))$ takes the form

$$\hat{\sigma}(s,t) = \hat{c}(s,t) + \hat{\sigma}^2(t)$$

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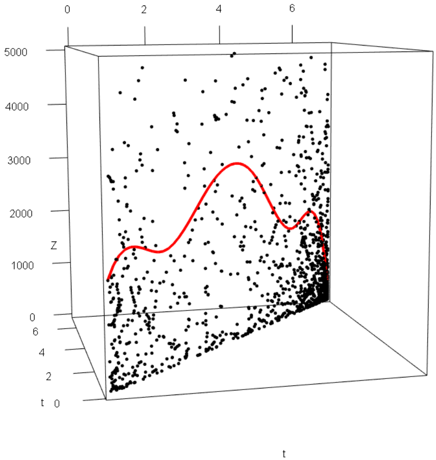
Auction Price



Variance Estimate $\hat{\sigma}^2$

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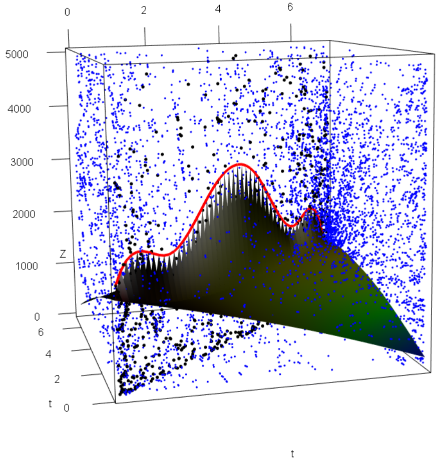
Auction Price



Covariance Estimate $\hat{\sigma}(s,t) = \hat{c}(s,t) + \hat{\sigma}^2(t)$

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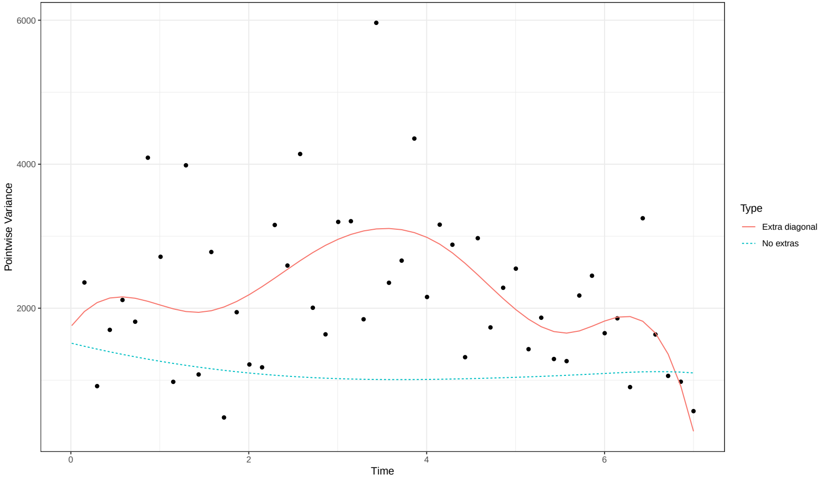
Auction Price



Comparison of Variance Estimates

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6. Let's consider the estimate of $\hat{\sigma}(s,t)$ and its Karhunen – Loève decomposition for functions

$$\hat{\sigma}(s,t) = \sum_{j=1}^{\infty} \lambda_j \xi_j(s) \xi_j(t) \quad \Rightarrow \text{ obtain } \hat{\xi}_j(t), \hat{\lambda}_j,\ j = 1, \ldots, K.$$

7. Estimate principal scores $c_{ij} = \int \xi_j(t)[Y_i(t) - \mu(t)]dt$ through the conditional expectation

$$\hat{c}_{ij} = \mathrm{E}[c_{ij}|\mathbf{Y}_i] = \hat{\lambda}_j \hat{\boldsymbol{\xi}}_j^T \hat{\boldsymbol{\Sigma}}_i^{-1}(\mathbf{Y}_i - \hat{\boldsymbol{\mu}}_i)$$

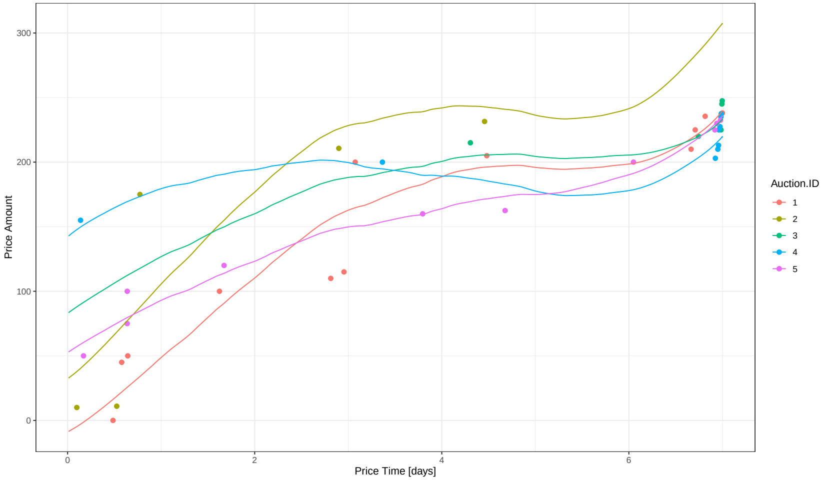
Yao et al. (2005)

8. Finally, reconstruct the whole curves

$$\widehat{Y}_i(t) = \hat{\mu}(t) + \sum_{j=1}^{K} \hat{c}_{ij} \hat{\xi}_j(t).$$

Filled Data

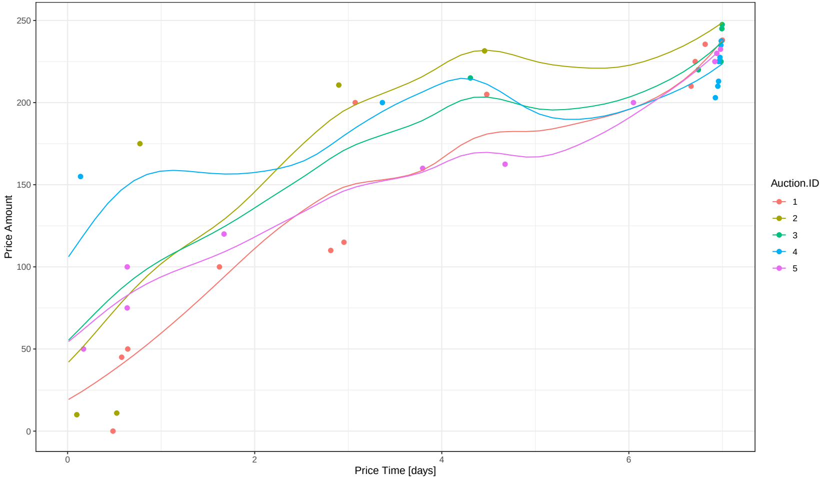
Auction Price – proposed method



Auction Prices Estimates

Filled Data

Auction Price – FDAPACE



Auction Prices Estimates

Kernel Smoothing

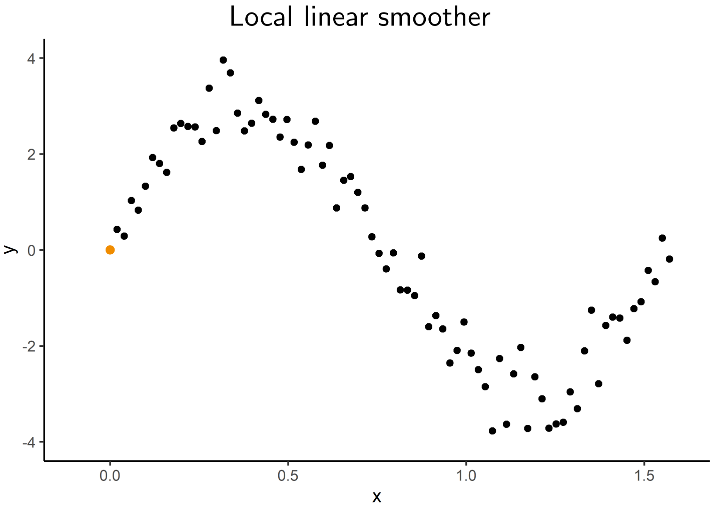
Mean function estimate

Local linear smoother with global bandwidth

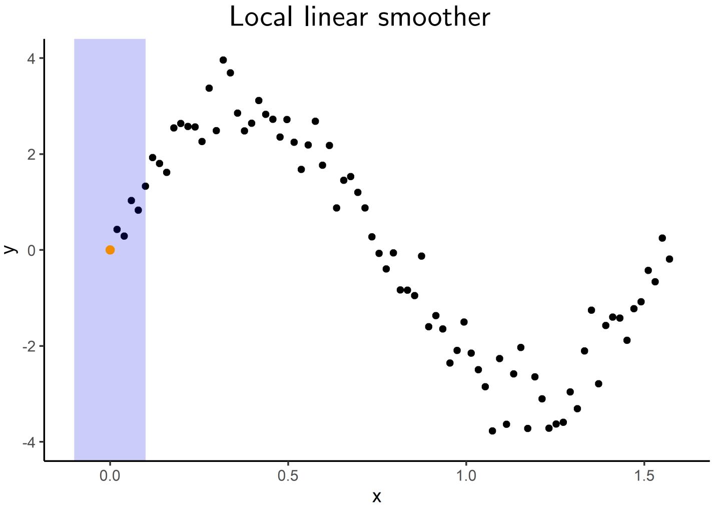
$$\sum_{i=1}^{n} \sum_{j=1}^{N_i} \left[K\left(\frac{T_{ij} - t}{h} \right) Y_{ij} - \beta_0 - \beta_1 (t - T_{ij}) \right]^2 \rightarrow \min$$

- $K(x)$ ... **kernel** function (a symmetric density)
- h ... **global** bandwidth
- $\hat{\mu}(t) = \hat{\beta}_0(t)$

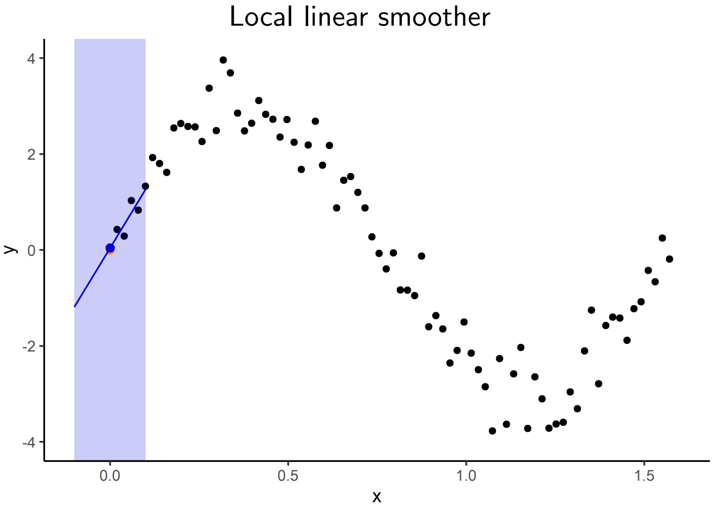
Kernel Smoothing



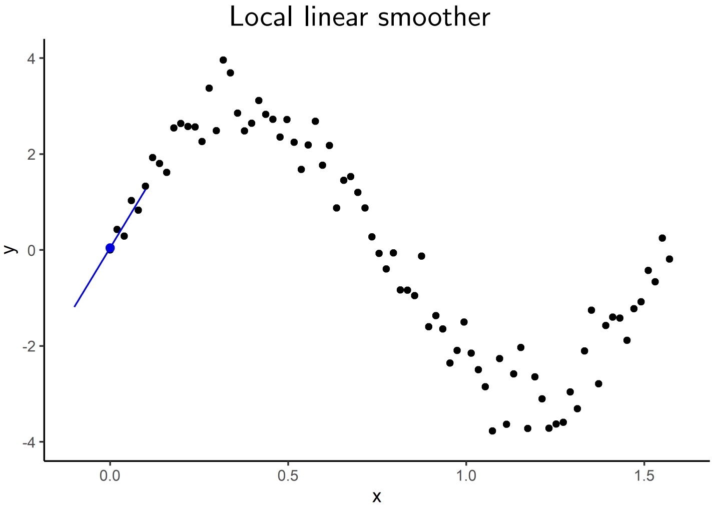
Kernel Smoothing



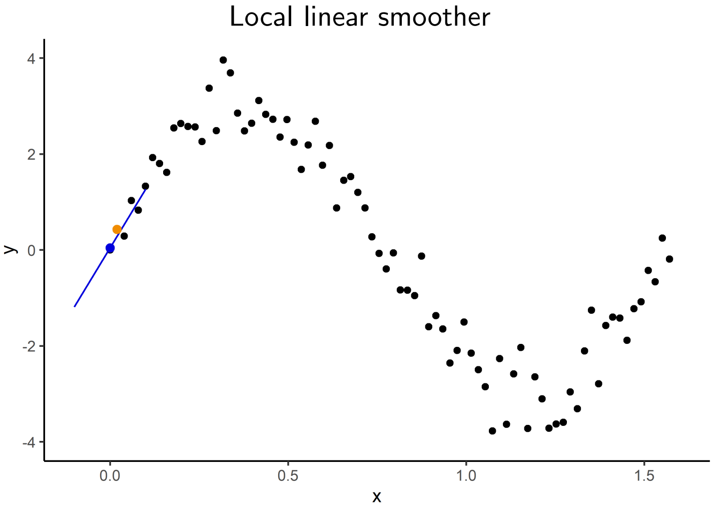
Kernel Smoothing



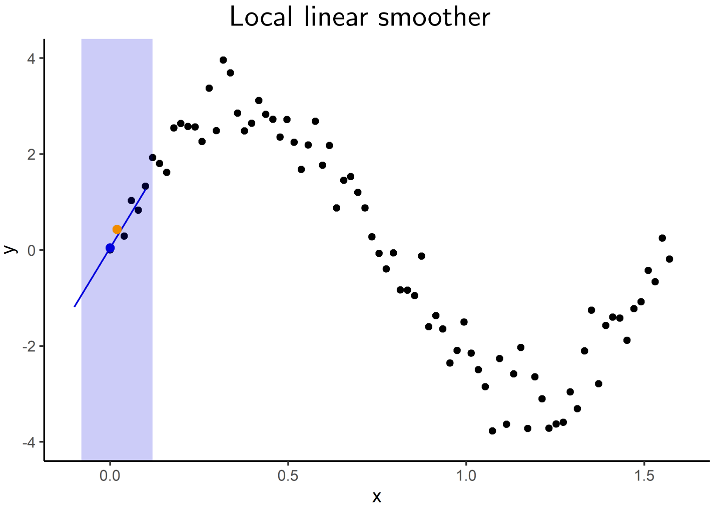
Kernel Smoothing



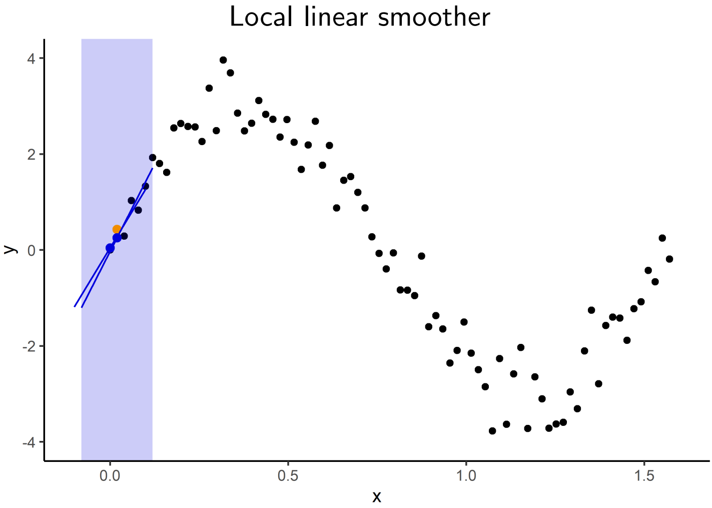
Kernel Smoothing



Kernel Smoothing



Kernel Smoothing



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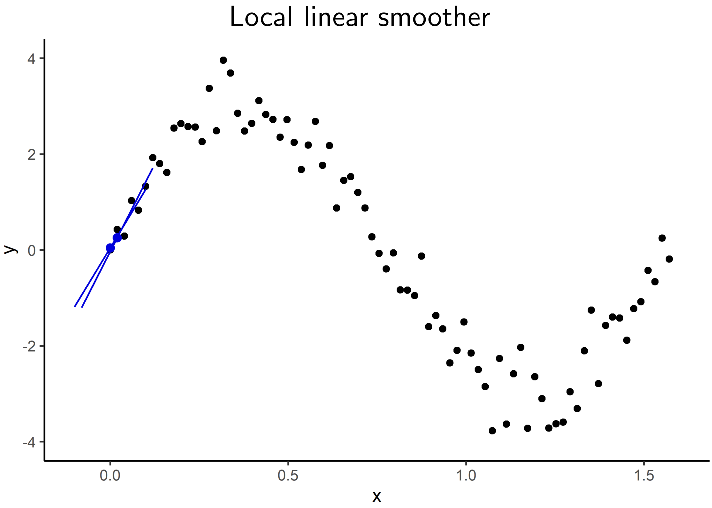

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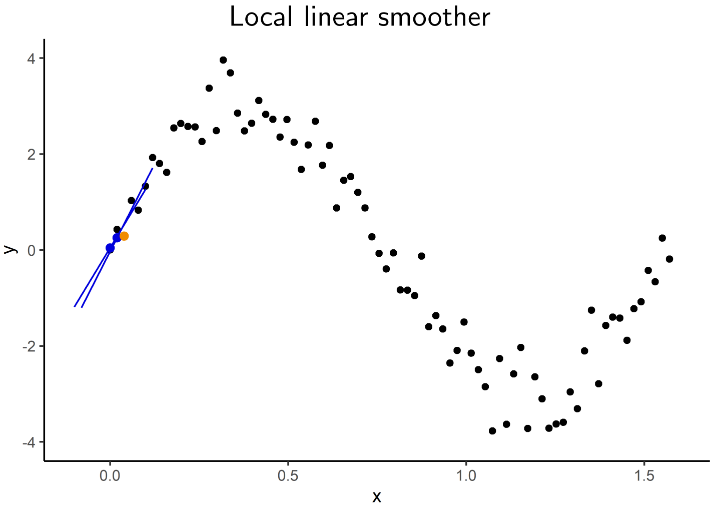
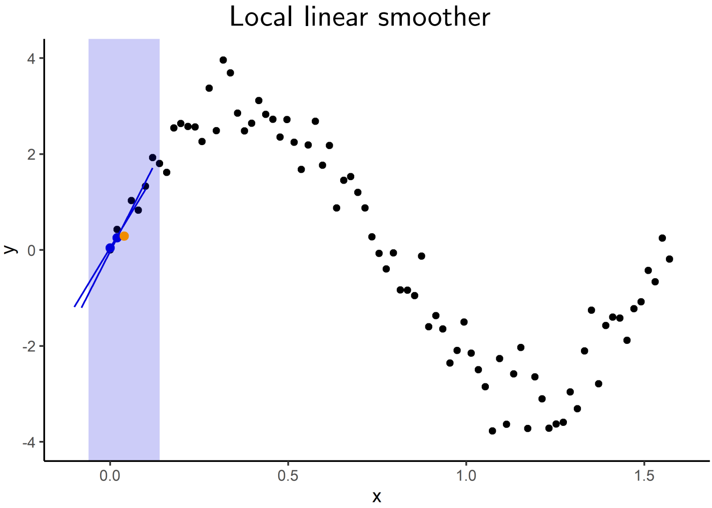
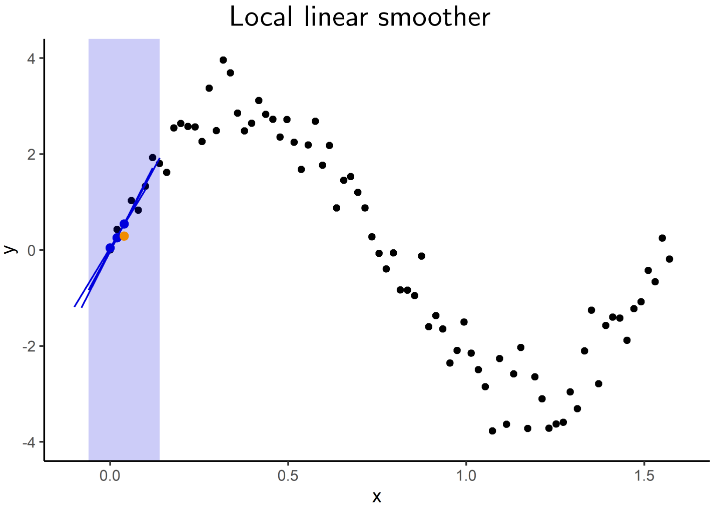


Kernel Smoothing



Kernel Smoothing



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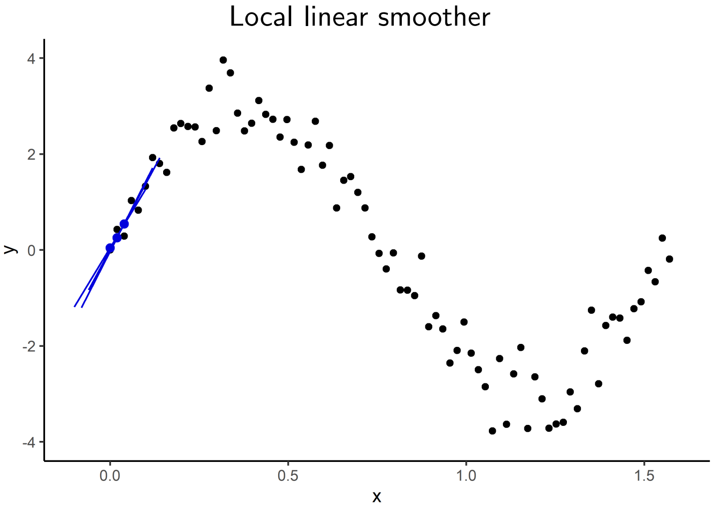

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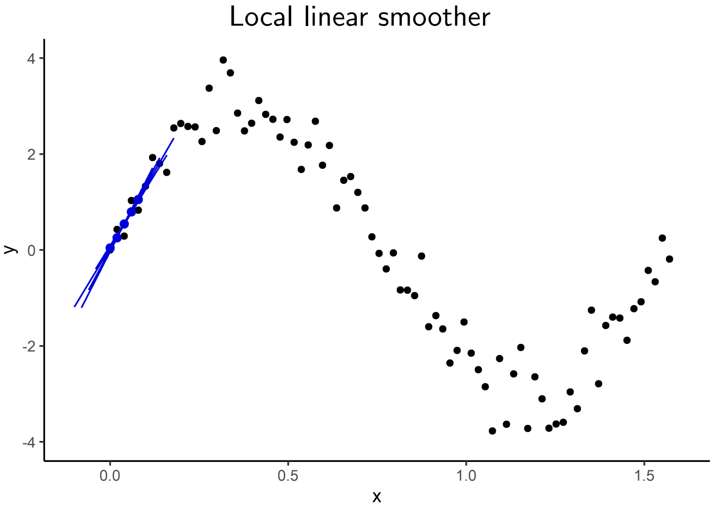

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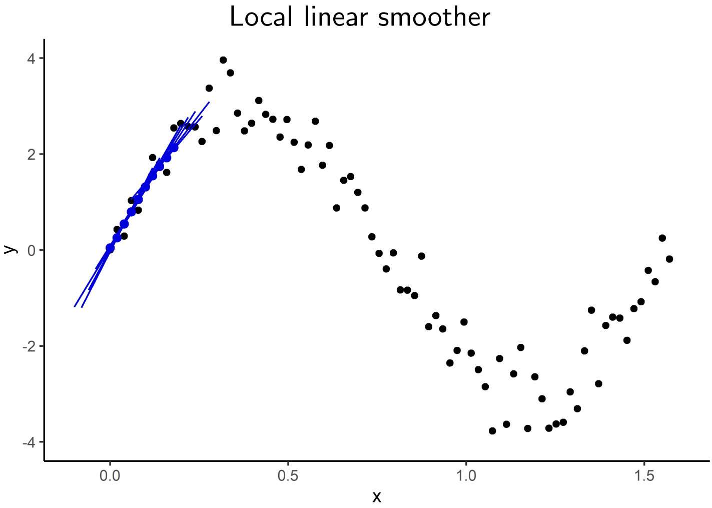

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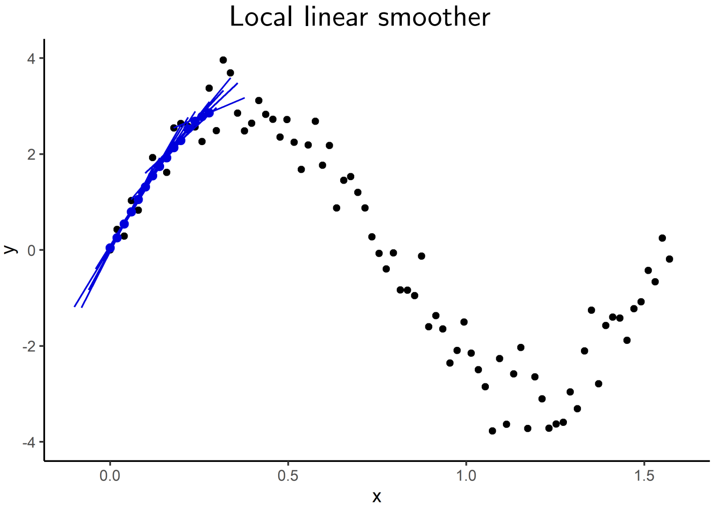

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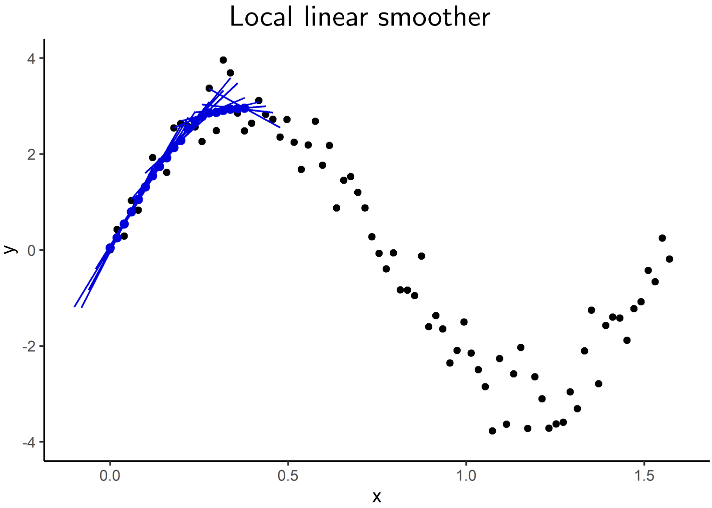

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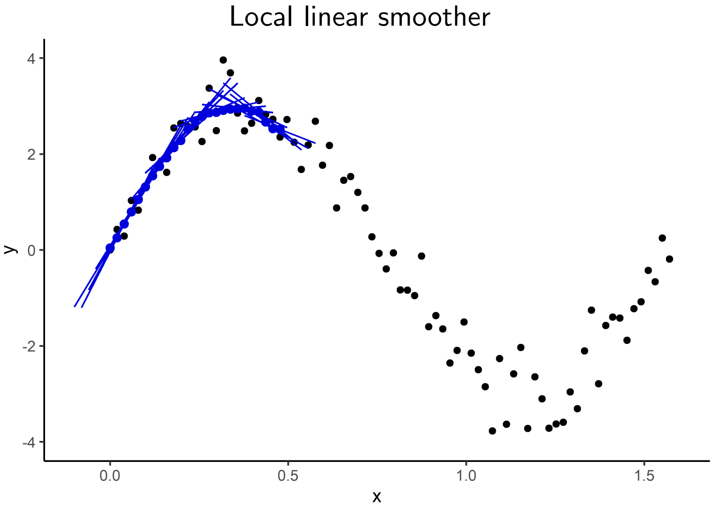

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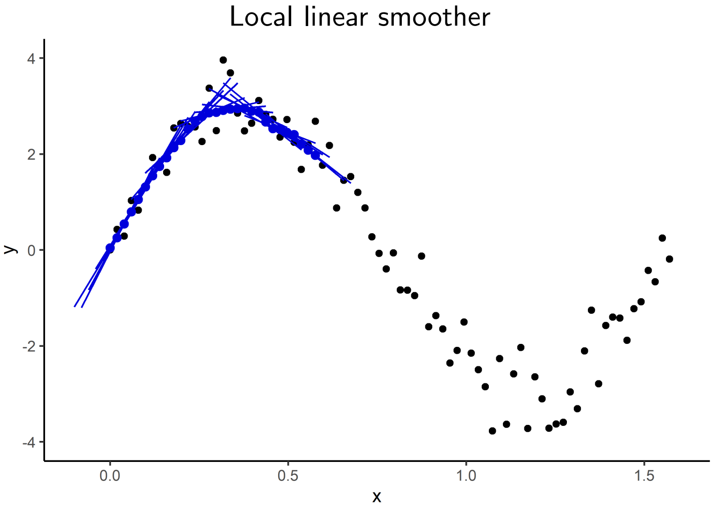

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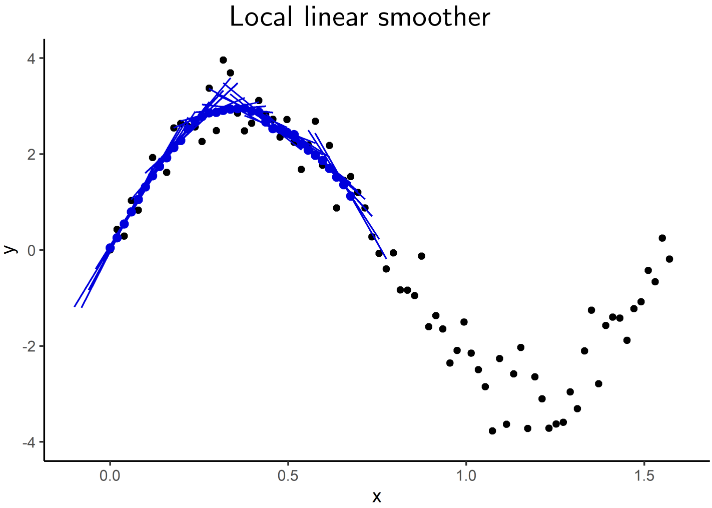

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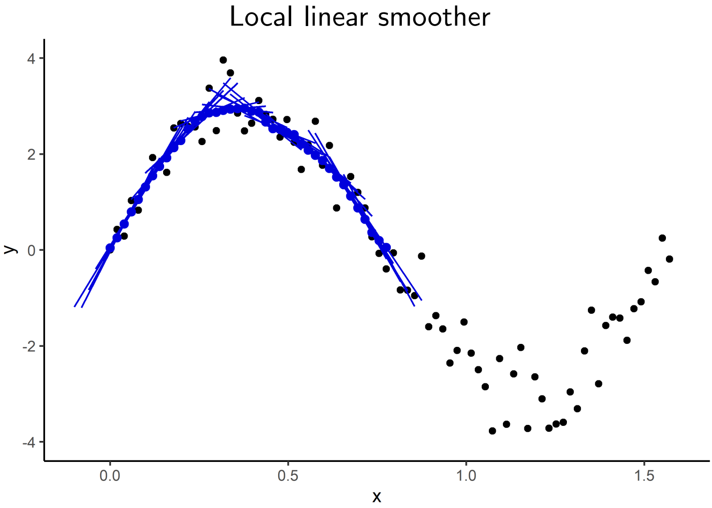
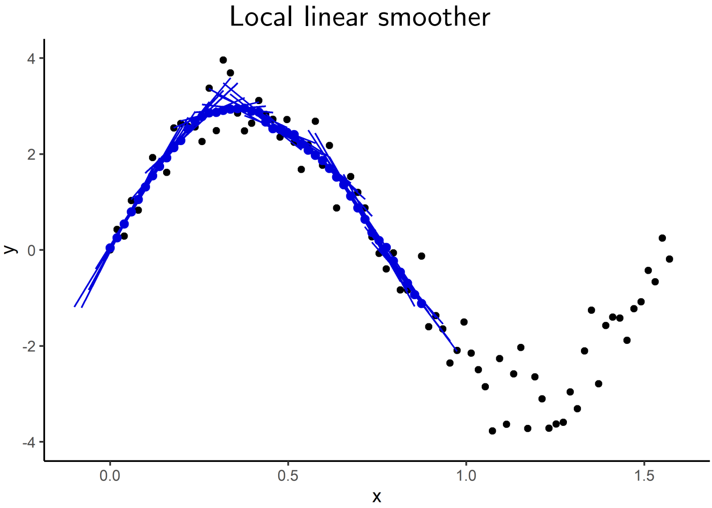
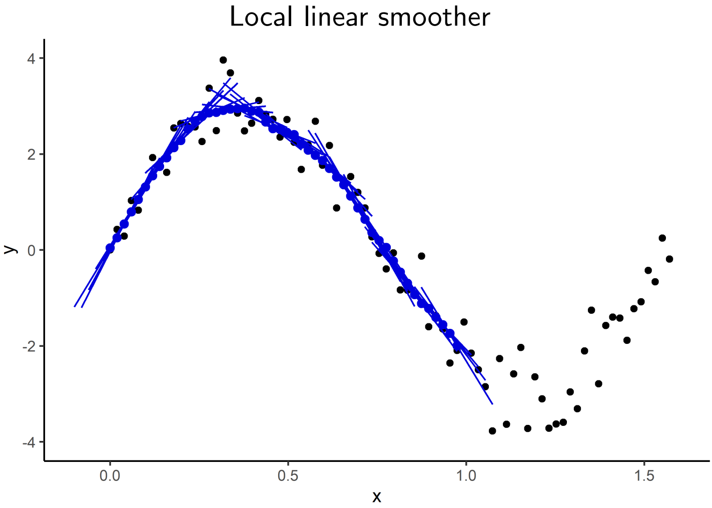


Kernel Smoothing



Kernel Smoothing



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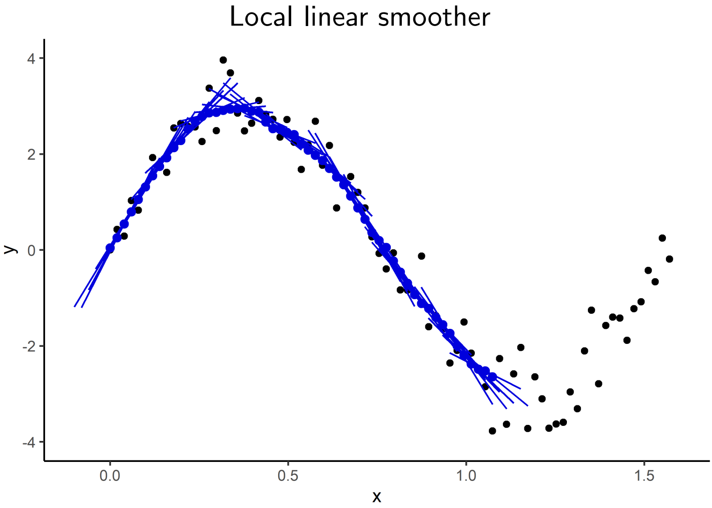

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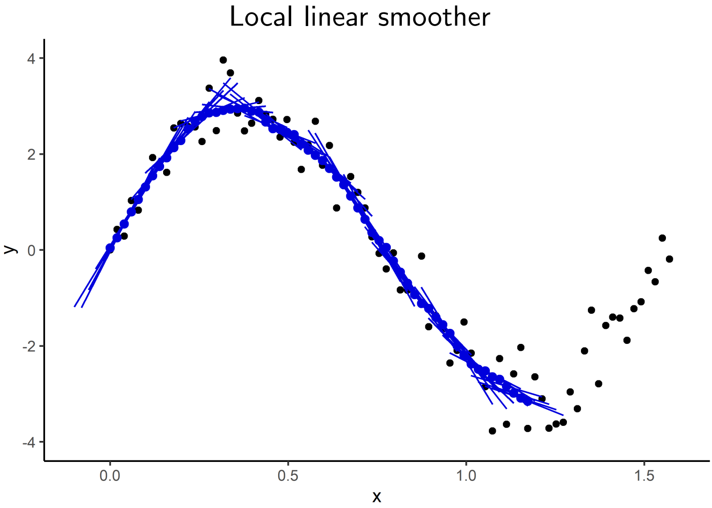

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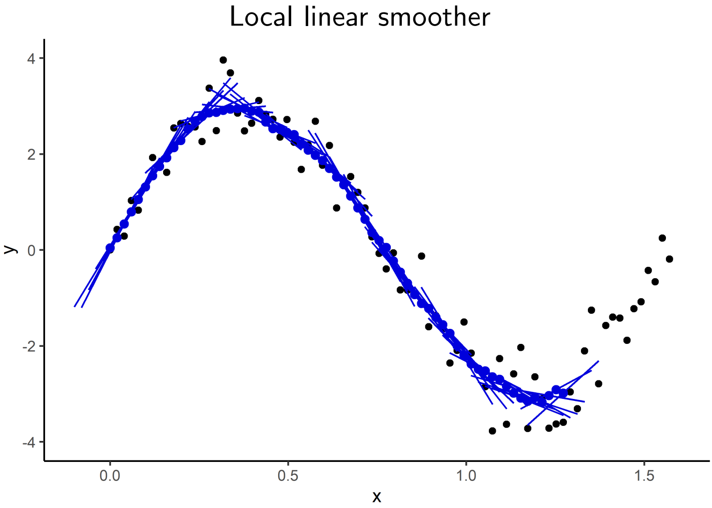

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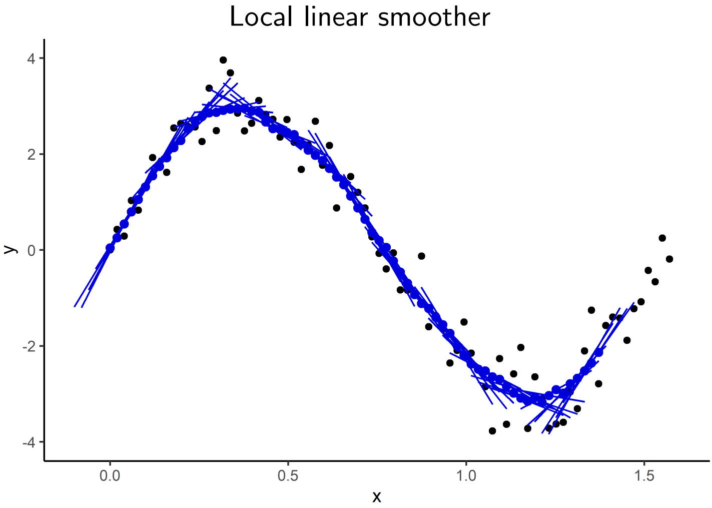

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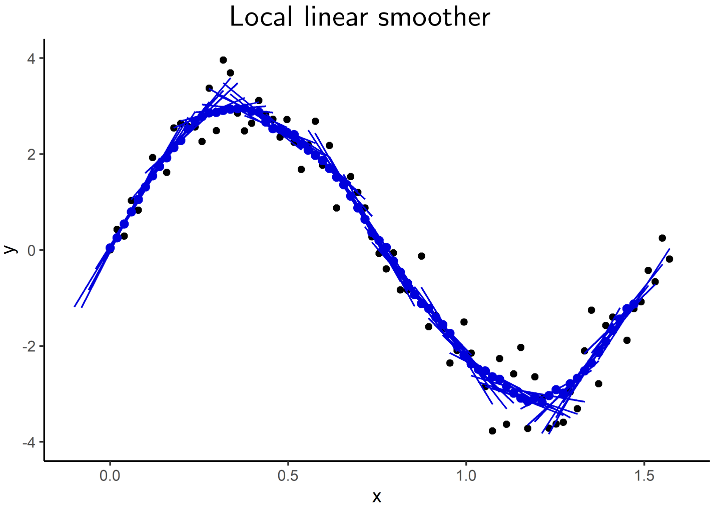

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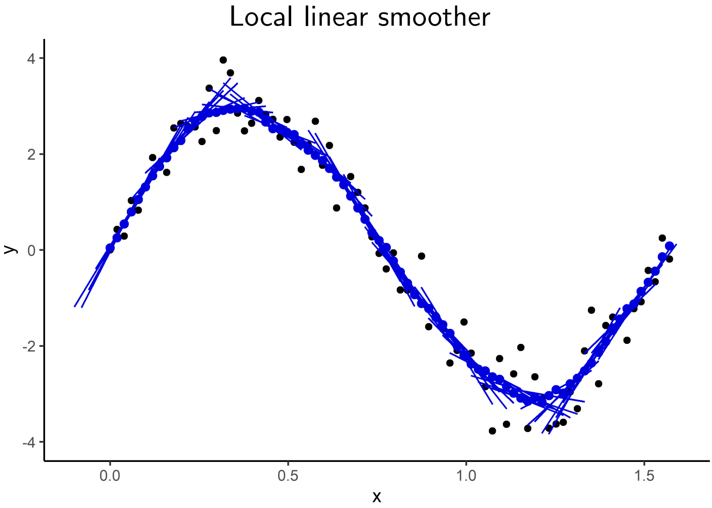

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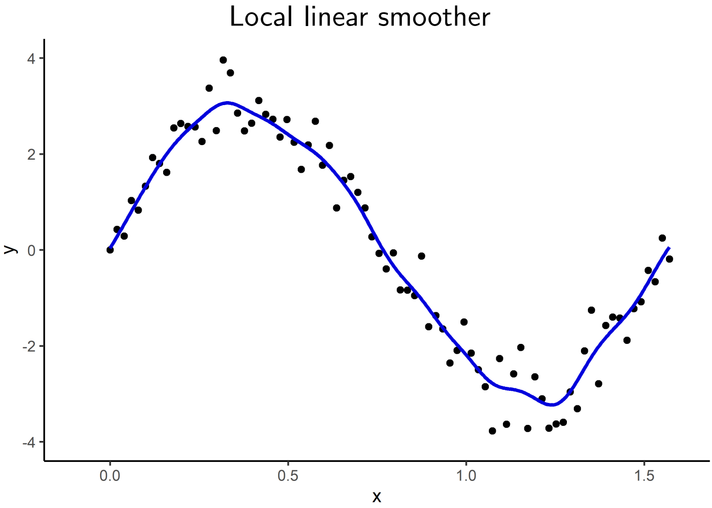

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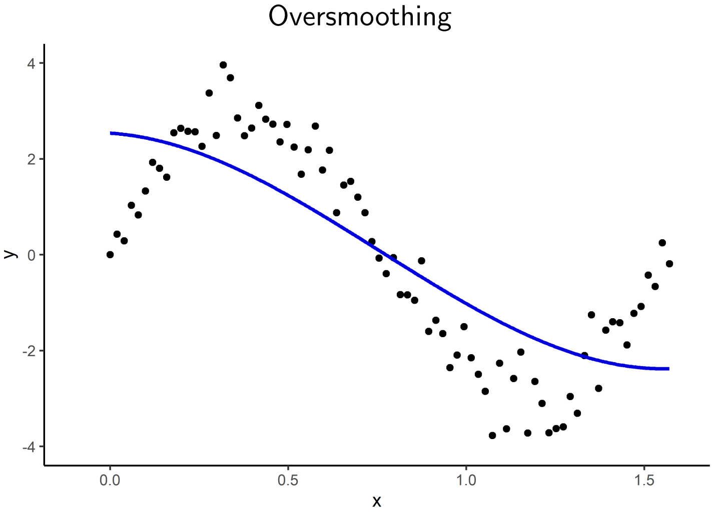

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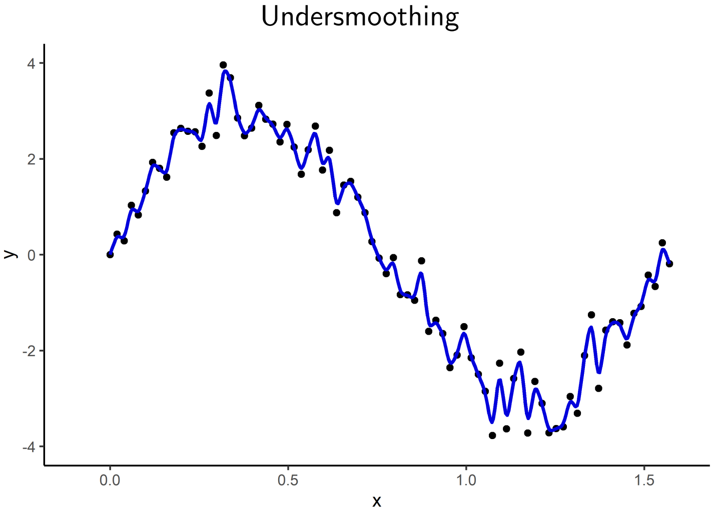


Kernel Smoothing

Mean function estimate

Local linear smoother with **global** bandwidth

$$\sum_{i=1}^{n}\sum_{j=1}^{N_i}\left[K\left(\frac{T_{ij}-t}{h}\right)Y_{ij}-\beta_0-\beta_1(x-T_{ij})\right]^2\to\min$$

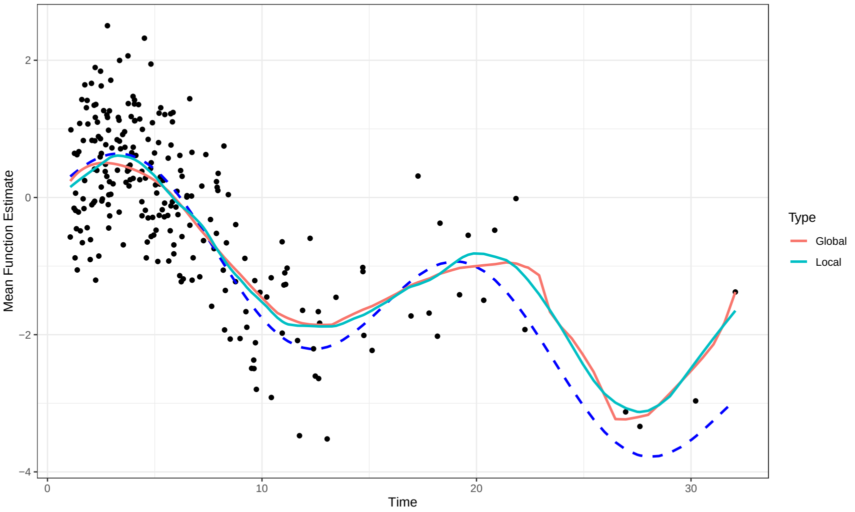
Local linear smoother with **local** bandwidth (Fan & Gijbels (1992))

$$\sum_{i=1}^{n}\sum_{j=1}^{N_i}\left[\alpha(T_{ij})K\left(\frac{T_{ij}-t}{h}\alpha(T_{ij})\right)Y_{ij}-\beta_0-\beta_1(t-T_{ij})\right]^2\to\min$$

optimal $\alpha(\cdot)\sim f^{1/5}(\cdot)$

Kernel Smoothing

Simulation



Kernel Smoothing

Covariance function estimate

Local linear smoother with global bandwidth

$$\sum_{i=1}^{n}\sum_{\substack{j_1=1\\ j_1\neq j_2}}^{N_i}\sum_{j_2=1}^{N_i}\Bigg[K\left(\frac{T_{ij_1}-s}{h},\frac{T_{ij_2}-t}{h}\right)Z(T_{ij_1},T_{ij_2})$$
$$-\beta_0-\beta_{11}(s-T_{ij_1})-\beta_{12}(t-T_{ij_2})\Bigg]^2\to\min$$

- $\hat{c}(s,t)=\hat{\beta}_0(s,t)$
- **goal**: adapt the local bandwidth method

Kernel Smoothing

Covariance function estimate

Local linear smoother with local bandwidth

$$\sum_{i=1}^{n} \sum_{\substack{j_1=1 \\ j_1 \neq j_2}}^{N_i} \sum_{j_2=1}^{N_i} \Big[\alpha(T_{ij_1})\alpha(T_{ij_2}) K\left(\frac{T_{ij_1}-s}{h}\alpha(T_{ij_1}), \frac{T_{ij_2}-t}{h}\alpha(T_{ij_2}) \right) Z(T_{ij_1}, T_{ij_2})$$

$$-\beta_0 - \beta_{11}(s - T_{ij_1}) - \beta_{12}(t - T_{ij_2})\Big]^2 \to \min$$

Known issues:

- symmetry of $\hat{c}(s,t)$ (OK for symmetric kernels)
- positive definiteness of $\hat{c}(s,t)$ (particularly depends on h)
- optimal $\alpha(\cdot)$
- optimal h

Problems to solve

1. Motor Oil Data
   The dataset contains amount of Fe particles depending on operating time and a number of oil changes. Data were collected 2006 – 2016 from 29 heavy-duty army vehicles.
   - Load the variable `df.motor` from the `motoroil.RData` file and plot it (see Figure 1).
   - Use functions from the file `functionsM7777.R` to fill the data (see Figure 2).
   - Try to neglect the number of oil changes and put all groups together (see Figure 3).
   - Fill the data using one of mentioned methods (see Figure 4).
   - Do the same with using the `FPCA` package and compare results (see Figures 5, 6).
   - (optional) Is the number of oil changes negligible? Conduct the fANOVA analysis. Is it correct to do it?

Problems to solve

Motor Oil Data

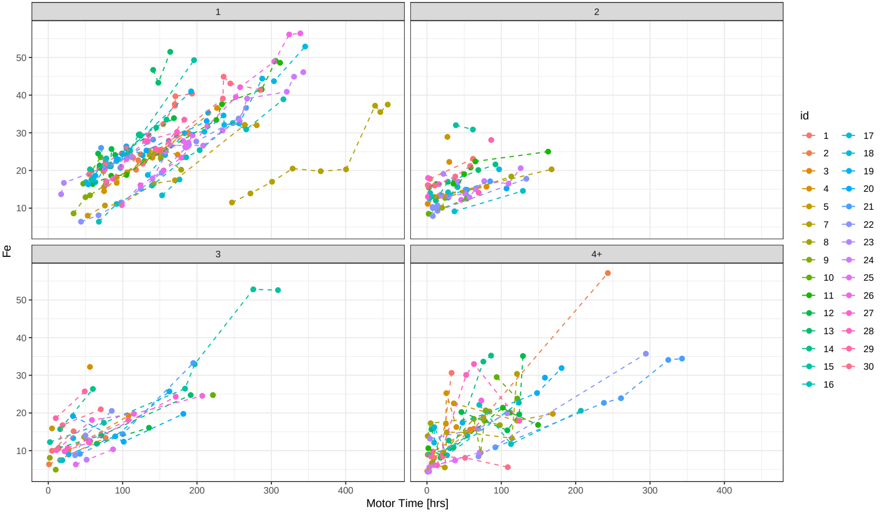


Figure 1.

Motor Oil Data – filled

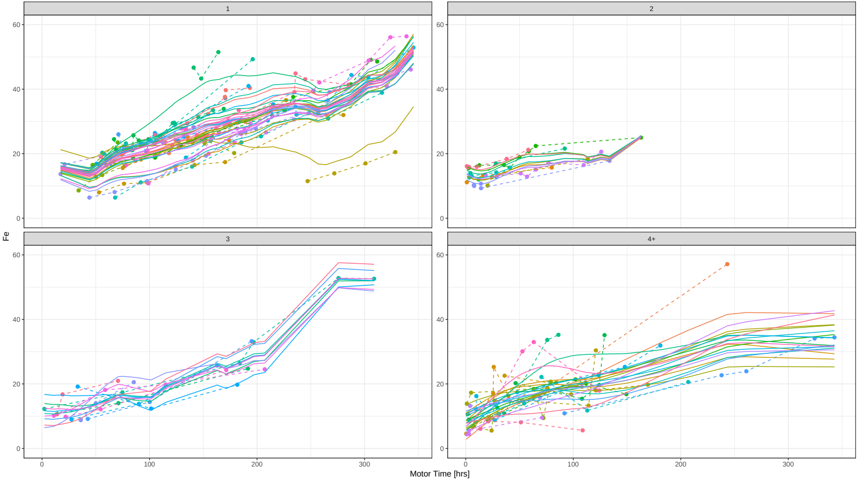


Figure 2.

Problems to solve

Motor Oil Data

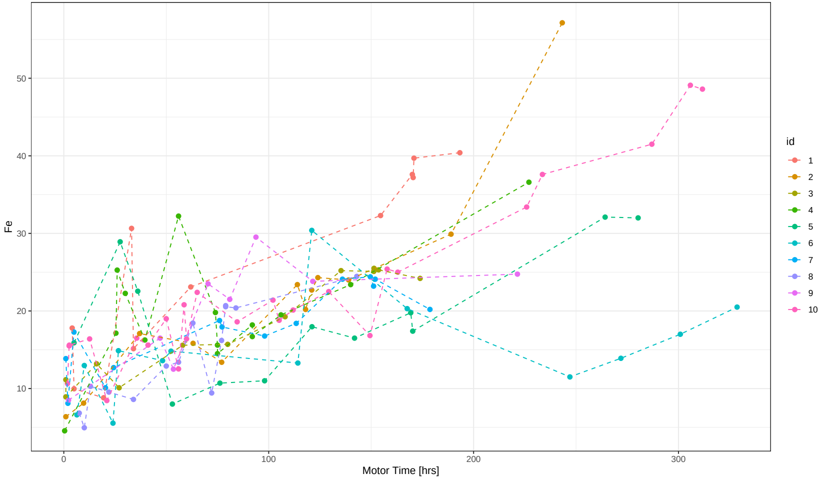


Figure 3.

Motor Oil Data – filled

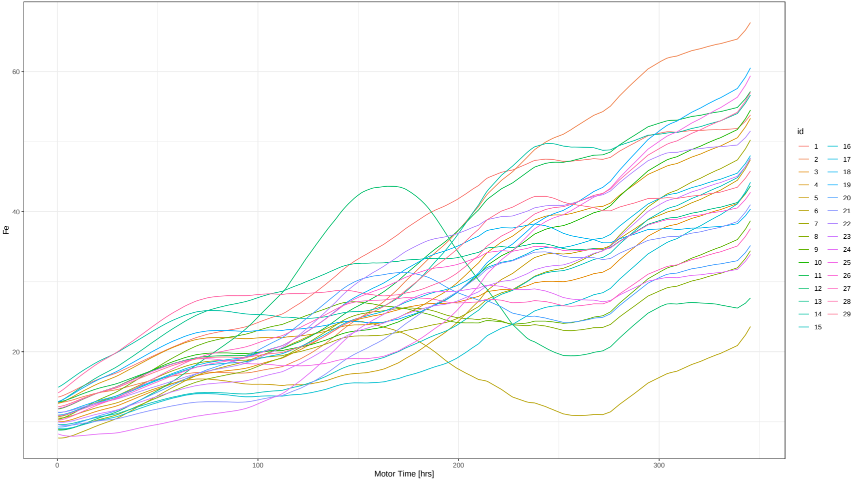


Figure 4.

Problems to solve

functionsM777.R

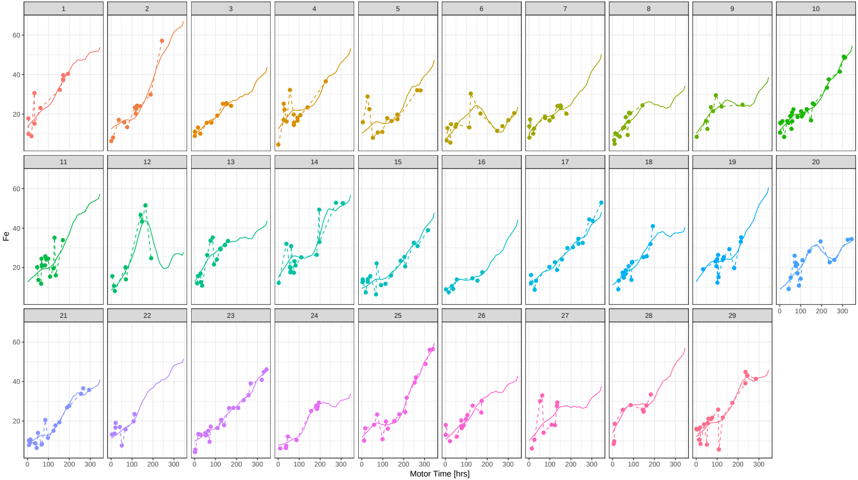


Figure 5.

Problems to solve

FDAPACE

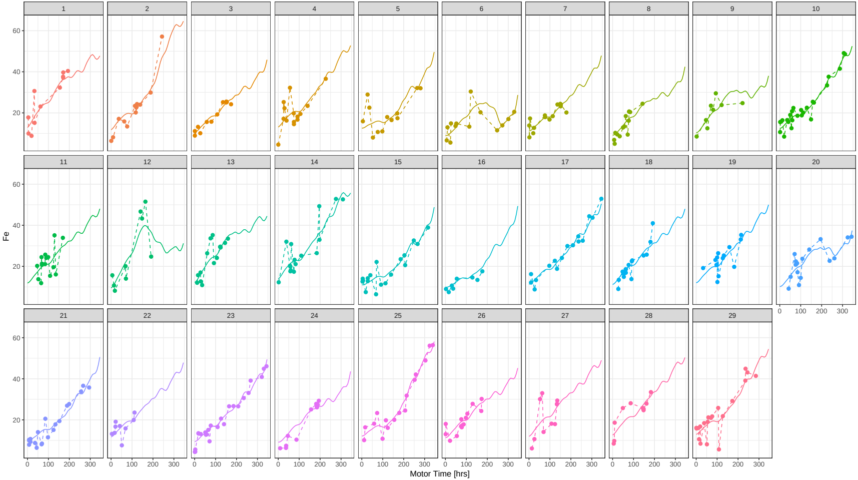


Figure 6.