INTRODUCTION TO ALGEBRAIC TOPOLOGY

MARTIN ČADEK

1. Basic notions and constructions

1.1. Notation. The closure, the interior and the boundary of a topological space X will be denoted by $\overline{X}$, $\operatorname{int} X$ and ∂X, respectively. The letter I will stand for the interval $[0, 1]$. $\mathbb{R}^n$ and $\mathbb{C}^n$ will denote the vector spaces of n-tuples of real and complex numbers, respectively, with the standard norm $\|x\| = \sum_{i=1}^{n} |x_i|^2$. The sets

$$D^n = \{x \in \mathbb{R}^n;\ \|x\| \leq 1\},$$
$$S^n = \{x \in \mathbb{R}^{n+1};\ \|x\| = 1\}$$

are the n-dimensional disc and the n-dimensional sphere, respectively.

1.2. Categories of topological spaces. Every category consists of objects and morphisms between them. Morphisms $f : A \to B$ and $g : B \to C$ can be composed in a morphism $g \circ f : A \to C$ and for every object B there is a morphism $\mathrm{id}_B : B \to B$ such that $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$.

The category with topological spaces as objects and continuous maps as morphisms will be denoted Top. Topological spaces with distinquished points (usually denoted by $*$) and continuous maps $f : (X, *) \to (Y, *)$ such that $f(*) = *$ form the category Top_*. Topological spaces X, A will be called *a pair of topological spaces* if A is a subspace of X (notation (X, A)). The notation $f : (X, A) \to (Y, B)$ means that $f : X \to Y$ is a continuous map which preserves subspaces, i. e. $f(A) \subseteq B$. The category Top^2 consists of pairs of topological spaces as objects and continuous maps $f : (X, A) \to (Y, B)$ as morphisms. Finally, Top^2_* will denote the category of pairs of topological spaces with distinquished points in subspaces and continuous maps preserving both subspaces and distinquished points.

The right category for doing algebraic topology is the category of compactly generated spaces. We will not go into details and refer to Chapter 5 of [May]. In fact, the majority of spaces we deal with in this text are compactly generated.

From now on, a space will mean a topological space and a map will mean a continuous map.

1.3. Homotopy. Maps $f, g : X \to Y$ are called *homotopic*, notation $f \sim g$, if there is a map $h : X \times I \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. This map is called *homotopy* between f and g. The relation $\sim$ is an equivalence. Homotopies in categories Top_*, Top^2 or Top^2_* have to preserve distinquished points, i. e. $h(*, t) = *$, subsets or both subsets and distinquished points, respectively.

Spaces X and Y are called *homotopy equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. We also say that the spaces X and Y have the same *homotopy type*. The maps f and g are called *homotopy equivalences*.

A space is called *contractible* if it is homotopy equivalent to a point.

Example. S^n and $\mathbb{R}^{n+1} - \{0\}$ are homotopy equivalent. As homotopy equivalences take the inclusion $f : S^n \to \mathbb{R}^{n+1} - \{0\}$ and $g : \mathbb{R}^{n+1} - \{0\} \to S^n$, $g(x) = x/\|x\|$.

1.4. Retracts and deformation retracts. Let $i : A \hookrightarrow X$ be an inclusion. We say that A is a *retract* of X if there is a map $r : X \to A$ such that $r \circ i = \mathrm{id}_A$. The map r is called a *retraction*.

We say that A is a *deformation retract* of X (sometimes also strong deformation retract) if $i \circ r : X \to A \to X$ is homotopic to the identity on X relative to A, i.e. there is a homotopy $h : X \times I \to X$ such that $h(-, 0) = \mathrm{id}_X$, $h(-, 1) = i \circ r$ and $h(i(-), t) = \mathrm{id}_A$ for all $t \in I$. The map h is called a *deformation retraction* .

Exercise A. Show that deformation retract of X is homotopy equivalent to X.

1.5. Basic constructions in Top. Consider a topological space X with an equivalence $\simeq$. Then $X/\simeq$ is the set of equivalence classes with the topology determined by the projection $p : X \to X/\simeq$ in the following way: $U \subseteq X/\simeq$ is open iff $p^{-1}(U)$ is open in X.

Exercise A. The map $f : (X/\simeq) \to Y$ is continuous iff the composition $f \circ p : X \to (X/\simeq) \to Y$ is continuous.

We will show this constructions in several special cases. Let A be a subspace of X. The *quotient* X/A is the space $X/\simeq$ where $x \simeq y$ iff $x = y$ or both x and y are elements of A. This space is often considered as a based space with base point determined by A. If $A = \emptyset$ we put $X/\emptyset = X \cup \{*\}$.

Exercise B. Prove that D^n/S^{n-1} is homeomorphic to S^n. For it consider $f : D^n \to S^n$

$$f(x_1, x_2, \ldots, x_n) = (2\sqrt{1 - \|x\|^2}x, 2\|x\|^2 - 1).$$

Disjoint union of spaces X and Y will be denoted $X \sqcup Y$. Open sets are unions of open sets in X and in Y. Let A be a subspace of X and let $f : A \to Y$ be a map. Then $X \cup_f Y$ is the space $(X \sqcup Y)/\simeq$ where the equivalence is generated by relations $a \simeq f(a)$.

The *mapping cylinder* of a map $f : X \to Y$ is the space

$$M_f = X \times I \cup_{f \times 1} Y$$

which arises from $X \times I$ and Y after identification of points $(x, 1) \in X \times I$ and $f(x) \in Y$.

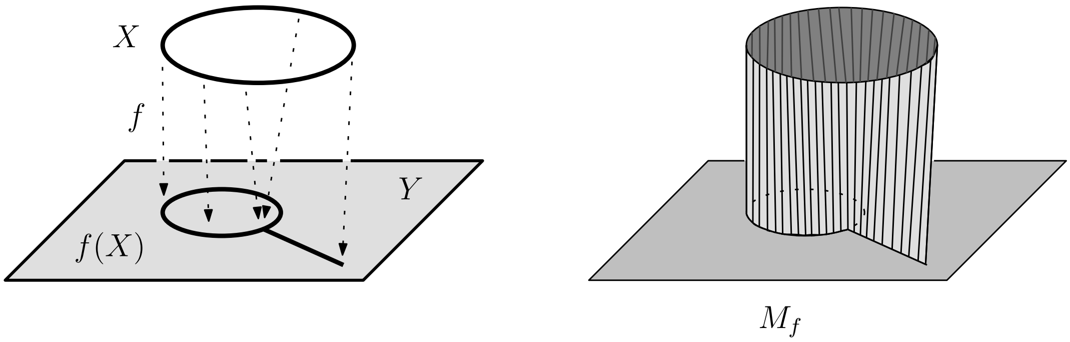

FIGURE 1.1. Mapping cylinder

Exercise C. We have two inclusions $i_X : X = X \times \{0\} \hookrightarrow M_f$ and $i_Y : Y \hookrightarrow M_f$ and a retraction $r : M_f \to Y$. How is r defined?

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, "i_X"] \arrow[dr, "f"] & \\
Y \arrow[r, "i_Y"'] & M_f \arrow[r, "r"'] & Y
\end{tikzcd}$$

Prove that

(1) Y is a deformation retract of M_f,
(2) $i_X \circ r = f$,
(3) $i_Y \circ f \sim i_X$.

The *mapping cone* of a mapping $f : X \to Y$ is the space

$$C_f = M_f/(X \times \{0\}).$$

A special case of a mapping cone is the *cone* of a space X

$$CX = X \times I/(X \times \{0\}) = C_{\mathrm{id}_X}.$$

The *suspension* of a space X is the space

$$SX = CX/(X \times \{1\}).$$

Exercise D. Show that $SS^n = S^{n+1}$. For it consider the map $f : S^n \times I \to S^{n+1}$

$$f(x,t) = (\sqrt{1-(2t-1)^2}x, 2t-1).$$

The *join* of spaces X and Y is the space

$$X \star Y = X \times Y \times I/\simeq$$

where $\simeq$ is the equivalence generated by $(x, y_1, 0) \simeq (x, y_2, 0)$ and $(x_1, y, 1) \simeq (x_2, y, 1)$.

Exercise E. Show that the join operation is associative and compute the joins of two points, two intervals, several points, $S^0 \star X$, $S^n \star S^m$.

1.6. Basic constructions in Top_* and Top^2. Let X be a space with a base point x_0. The *reduced suspension* of X is the space

$$\Sigma X = SX/(\{x_0\} \times I)$$

with base point determined by $x_0 \times I$. In the next section in **??** we will show that ΣX is homotopy equivalent to SX.

The space

$$(X, x_0) \vee (Y, y_0) = X \times \{y_0\} \cup \{x_0\} \times Y$$

with distinquished point (x_0, y_0) is called the *wedge* of X and Y and usually denoted only as $X \vee Y$.

The *smash product* of spaces (X, x_0) and (Y, y_0) is the space

$$X \wedge Y = X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y) = X \times Y/X \vee Y.$$

Analogously, the smash product of pairs (X, A) and (Y, B) is the pair

$$(X \times Y, A \times Y \cup X \times B).$$

Exercise A. Show that $S^m \wedge S^n = S^{n+m}$. One way how to do it is to prove that

$$X/A \wedge Y/B \cong X \times Y/A \times Y \cup X \times B.$$

1.7. Homotopy extension property. We say that a pair of topological spaces (X, A) has the *homotopy extension property* (abbreviation HEP) if any map $f : X \to Y$ and any homotopy $h : A \times I \to Y$ such that $h(a, 0) = f(a)$ for $a \in A$, and

$$f \cup h : X \times \{0\} \cup A \times I \to Y$$

is continuous, can be extended to a homotopy $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(a, t) = h(a, t)$ for all $x \in X$, $a \in A$ and $t \in I$, i.e. H is an arrow making the diagram

$$\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r, "f \cup h"] \arrow[d] & Y \\
X \times I \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

commutative. If the pair (X, A) satisfies HEP, we call the inclusion $A \hookrightarrow X$ a *cofibration*.

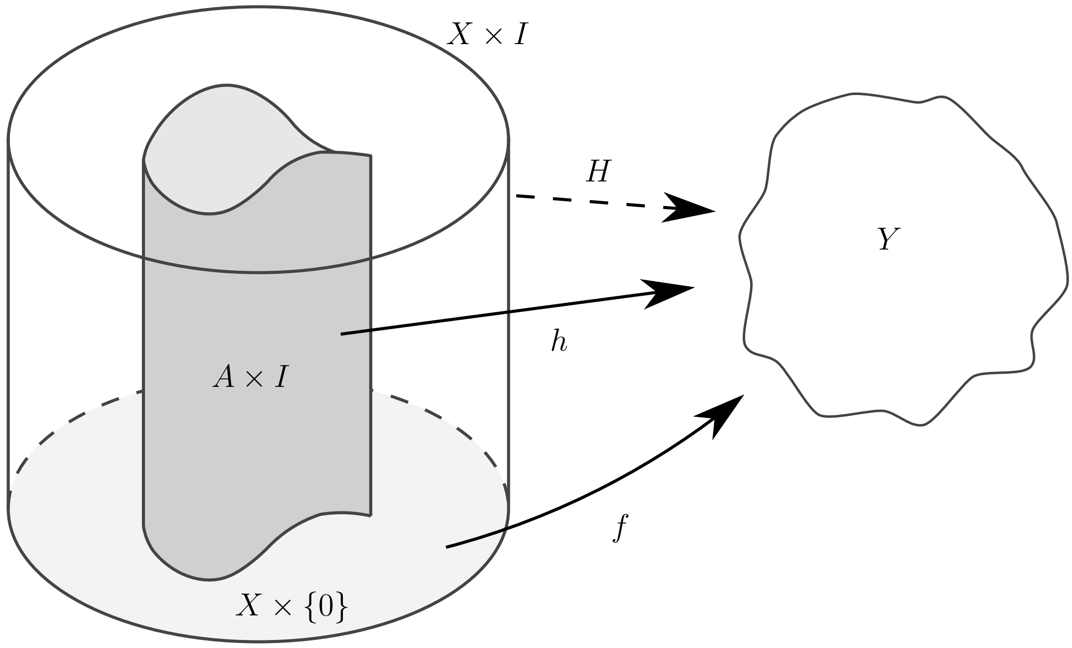

FIGURE 1.2. Homotopy extension property

Theorem. *A pair (X, A) has HEP if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.*

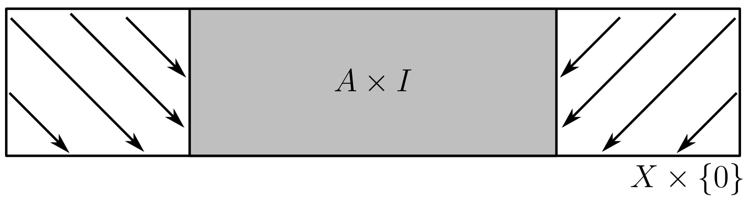

FIGURE 1.3. Retraction $X \times I \to X \times \{0\} \cup A \times I$

Exercise A. Using this Theorem show that the pair (D^n, S^{n-1}) satisfies HEP. Many other examples will be given in the next section.

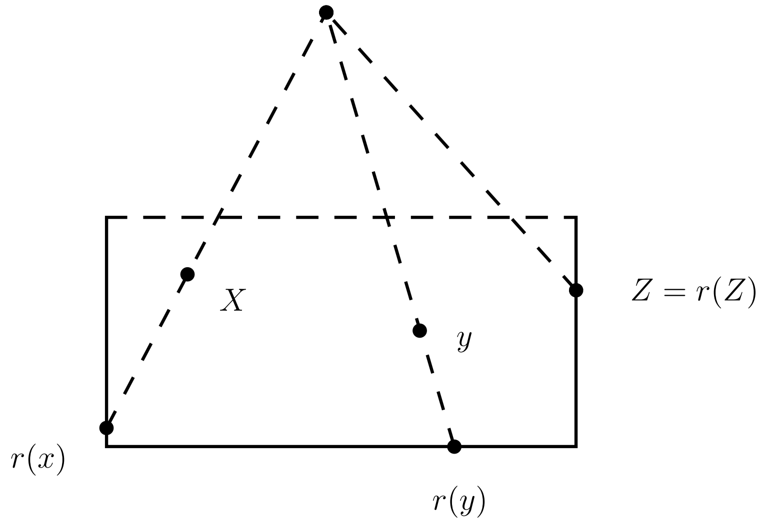

FIGURE 1.4. Retraction $D^1 \times I \to D^1 \times \{0\} \cup S^0 \times I$

Proof of Theorem. Let (X, A) has HEP. Put $Y = X \times \{0\} \cup A \times I$ and consider $f \cup h : X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$ to be an identity. Its extension $H : X \times I \to X \times \{0\} \cup A \times I$ is a retraction.

Let $r : X \times I \to X \times \{0\} \cup A \times I$ be a retraction. Given a map f and a homotopy h as in the definition which together determine a continuous map $F = (f \cup h) : X \times \{0\} \cup A \times I \to Y$, then $H = F \circ r$ is an extension of $f \cup h$. □

Exercise B. Let a pair (X, A) satisfy HEP and consider a map $g : A \to Y$. Prove that $(X \cup_g Y, Y)$ also satisfies HEP.

Exercise C. Let X be a Hausdorff compact space and let an inclusion $A \hookrightarrow X$ is a cofibration. Prove that A is a closed subset of X.

Exercise D. Consider the closed subset set $A = \{1/n \in \mathbb{R}; n = 0, 1, 2, \dots\} \cup \{0\}$ of the interval $[0, 1]$. However, the inclusion $A \hookrightarrow [0, 1]$ is not a cofibration. Prove it.

Exercise E. Let M_f be a mapping cylinder of a map $f : X \to Y$. Show that the inclusion $i_X : X \hookrightarrow M_f$ is a cofibration. In particular, the map $f : X \to Y$ can be factored into the composition $r \circ i_X$ of the cofibration i_X and the homotopy equivalence r. (See the exercise after the definition of the mapping cylinder.)

CZ.1.07/2.2.00/28.0041
Centrum interaktivních a multimediálních studijních opor pro inovaci výuky a efektivní učení

evropský sociální fond v ČR
EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
OP Vzdělávání pro konkurenceschopnost
INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ