Exercise 63. *Using homotopy groups show that $\mathbb{R}P^k$ is not a retract of $\mathbb{R}P^n$, $n > k \geq 1$.*

Solution. Retract: The composition $\mathbb{R}P^k \overset{i}{\hookrightarrow} \mathbb{R}P^n \xrightarrow{r} \mathbb{R}P^k$ is the identity, so we have the following diagram and want to use it to obtain a contradiction.

$$\begin{tikzcd}
\pi_*(\mathbb{R}P^k) \arrow[r, "\pi_*(i)"] \arrow[dr, "\mathrm{id}"'] & \pi_*(\mathbb{R}P^n) \arrow[d, "\pi_*(r)"] \\
& \pi_*(\mathbb{R}P^k)
\end{tikzcd}$$

From the long exact sequence of the fibration, we know that $0 = \pi_1(S^k) \to \pi_1(\mathbb{R}P^k) \xrightarrow{\cong} \pi_0(S^0) \to 0$ and $\pi_i(S^k) = \pi_i(\mathbb{R}P^k)$ for $k \geq 2$. Then considering diagram

$$\begin{tikzcd}
\pi_k(S^k) = \mathbb{Z} \arrow[r] \arrow[dr] & \pi_k(S^n) = 0 \arrow[d] \\
& \pi_k(S^k) = \mathbb{Z}
\end{tikzcd}$$

we see that we are factoring identity through zero group and that is Mission Impossible: Factor Zero (in theaters never). Continue with $k = 1$. We use the knowledge of $\mathbb{R}P^1 \cong S^1$. Then our diagram is a triangle with $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}$ with identity $\mathbb{Z} \to \mathbb{Z}$, factoring identity through finite group is Mission Impossible: Group protocol (in theaters maybe one day, one can only hope) ...and we are done. □

Exercise 64. *Consider the map $q\colon S^1 \times S^1 \times S^1 \to S^3$ defined as a map $S^1 \times S^1 \times S^1 \to D^3/S^2$ where D^3 is a small disk in the triple torus which is the identity in the interior of D^3 and constant on its complement. Further, consider the Hopf map $p\colon S^3 \to S^2 = \mathbb{C}P^1$ (described in Hopf fibration $S^1 \to S^3 \to S^2$). Compute q_* and $(pq)_*$ in homotopy groups. Show that pq is not homotopic to a constant map.*

Solution. Take following diagram and treat it like your own:

$$\begin{tikzcd}
& S^1 \arrow[d] & \text{(fiber)} \\
S^1 \times S^1 \times S^1 \arrow[r, "q"] & S^3 \arrow[d, "p"] & \subseteq \mathbb{C}^2 \\
& S^2 & = \mathbb{C}P^1
\end{tikzcd}$$

where $p(z_1, z_2) = \dfrac{z_1}{z_2}$. We know that only nontrivial homotopy group of the triple torus is $\pi_1(S^1 \times S^1 \times S^1)$, but $\pi_1 S^3$ is is trivial, so q_* is zero in homotopy groups and the composition as well. By the following diagram we have H, thanks to homotopy lifting property, of course. (denote $(S^1)^3$ the triple torus $S^1 \times S^1 \times S^1$)

$$\begin{tikzcd}
& S^1 = p^{-1}(s_0) \arrow[d] \\
(S^1)^3 \arrow[ur] \arrow[r, "q"] \arrow[d] & S^3 \arrow[d] \\
(S^1)^3 \times I \arrow[ur, "H"] \arrow[r, "h"'] & (S^2, s_0)
\end{tikzcd}$$

where $h(0,-)$ is constant map, $h(1,-) = pq$, $p(H(0)) = s_0$,$\operatorname{Im} H(0) \subseteq p^{-1}(s_0) = S^1$ and $H(1) = q \sim H(0)$. However, $H(0)_*$ and q_* differ in the third homology groups:

$$\begin{tikzcd}
& H_3(S^1) = 0 \arrow[d] \\
H_3((S^1)^3) \arrow[ur, "H(0)_*"] \arrow[r, "q_*=id"'] & H_3(S^3) = \mathbb{Z}
\end{tikzcd}$$

We are trying to factor through zero, a contradiction. □

Exercise 65. *(detail for the lecture 9. 5. 2017) If (X, A) is relative CW-complex such that there are no cells in dimension $\leq n$ in $X \setminus A$, then (X, A) is n-connected.*

Solution. Recall the definition of n-connectness of a pair. For $[f] \in \pi_i(X, A, x_0)$, $i \leq n$, use cell approximation of f: There is a cell map $q\colon (D^i, S^{i-1}, s_0) \to (X, A, x_0)$, such that $q \sim f$ relatively S^{i-1} and $q(D^i) \subseteq X^{(i)} = A$ since $X^{(-1)} = X^{(i)} = \cdots = X^{(n)} = A$. Note the following very useful criterion:

$[f] = 0$ in $\pi_i(X, A, x_0) \iff f \sim q$ relatively S^{i-1}, $g(D^i) = A$.

Thus $[f] = 0$ in our case, and we are done. □

Exercise 66. *Let $[X, Y]$ denote a set of homotopy classes of maps from X to Y. If (X, x_0) is a CW-complex and Y is path connected, then $[X, Y] \cong [(X, x_0), (Y, y_0)]$.*

Solution. Surely, $[(X, x_0), (Y, y_0)] \subseteq [X, Y]$ and denote the class in the left set as (g) and $[g]$ the class in the $[X, Y]$. The map $(g) \mapsto [g]$ is well defined and injective. To prove that it is also surjective take $[f] \in [X, Y]$. Using HEP for $f : X \times 0 \to Y$ and a curve $\omega : x_0 \times I \to Y$ which connects $f(x_0)$ with y_0 we get $g : X \times 1 \to Y$ such that $f \sim g$ and $g(x_0) = y_0$, then $[f] = [g]$ and $(g) \in [(X, x_0), (Y, y_0)]$. We are done. □

Exercise 67. *(application) We know that $deg(f)$ is an invariant of $[S^n, S^n] = \pi_n(S^n)$. Study $[S^{2n-1}, S^n] \cong \pi_{2n-1}(S^n)$ and describe its co called Hopf invariant $H(f)$.*

Solution. Have $f\colon \partial D^{2n} = S^{2n-1} \to S^n$ and $S^n \cup_f D^{2n}$. For $f \sim g$ we have $S^n \cup_f D^{2n} \simeq S^n \cup_g D^{2n}$, moreover $S^n \cup_f D^{2n} = C_f$ (the cylinder of f). For $n \geq 2$ we have $C_f = e^0 \cup e^n \cup e^{2n}$. Using cohomology: $H^*(C_f) = \mathbb{Z}$ for $* \in \{0, n, 2n\}$ and 0 elsewhere. Take $\alpha \in H^n(C_f)$ generator, we have cup product. Then $\alpha \cup \alpha \in H^{2n}(C_f)$ and for $\beta \in H^{2n}(C_f)$ we have $\alpha \cup \alpha = H(f)\beta$, where $H(f)$ is the Hopf invariant. □

Exercise 68. *(continuation of previous exercise) For n odd, what can we say in this case about Hopf inviariant? And for n even? Thanks.*

Solution. Knowing $\alpha \cup \beta = (-1)^{|\alpha||\beta|}\beta \cup \alpha$ we see that $\alpha \cup \alpha = 0$. So for n odd Hopf invariant is zero.

For n even consider the Hopf fibration $S^1 \to S^3 \to S^2 = \mathbb{C}P^1$. For $\mathbb{C}P^2 = D^4 \cup_f \mathbb{C}P^1$ (recall how $\mathbb{C}P^n$ is built up from $\mathbb{C}P^{n-1}$) we have $C_f = \mathbb{C}P^2$ and $H^*(\mathbb{C}P^2) = \mathbb{Z}[\alpha]/\langle\alpha^3\rangle$, with $\alpha \in H^2$. The generator of H^4 is α^2. We get that $H(f) = 1$. $\square$