Lecture 4 : Singular homology II

Recall the properties of sing. homology groups:

(1) Long exact sequence of a pair (X,A), $A \xhookrightarrow{i} X \xrightarrow{j} (X,A)$

$$H_n(A) \xrightarrow[i_*]{} H_n(X) \xrightarrow[j_*]{} H_n(X,A) \xrightarrow[\partial_*]{} H_{n-1}(A)$$

(2) Homotopy invariance $f, g : X \to Y$ homotopic means that $f_* = g_* : H_n(X) \to H_n(Y)$

(3) Excision: (A) $C \subset A \subset X$, $\overline{C} \subseteq \text{int } A$. Then the inclusion

$$(X \setminus C, A \setminus C) \hookrightarrow (X, A)$$

induces an isomorphism $H_n(X \setminus C, A \setminus C) \xrightarrow{\cong} H_n(X, A)$

(B) $A, B \subseteq X$, $X = \text{int } A \cup \text{int } B$. Then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces an iso

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

(4) Disjoint union $H_n\left(\bigsqcup_{\alpha \in A} X_\alpha\right) \cong \bigoplus_{\alpha \in A} H_n(X_\alpha)$

(5) Homology of a point $H_n(*) = \mathbb{Z}$ for $n = 0$, 0 otherwise

We show that excision theorems A and B are equivalent

(A) $\Rightarrow$ (B) Take $X = A \cup B$, $A = A$, $C = X \setminus B$. Then $\overline{C} = X \setminus \text{int } B \subseteq \text{int } A$ and

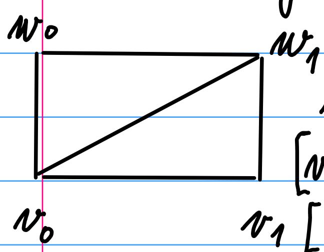
$$X \setminus C = B, \quad A \setminus (X \setminus B) = A \cap B$$

(B) $\Rightarrow$ (A) Take $B = X \setminus C$. Then $\text{int } B \cup \text{int } A = (X \setminus \overline{C}) \cup \text{int } A \supseteq (X \setminus \text{int } A) \cup \text{int } A = X$. Hence $B = X \setminus C$, $A \cap B = A \setminus C$.

Homotopy invariance - proof

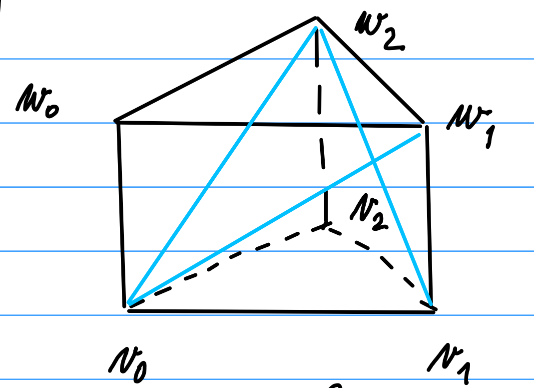
We show: if $f, g : X \to Y$ are homotopic, then $f_*, g_* : C_*(X) \to C_*(Y)$ are chain homotopic.

We start by dividing $\Delta^n \times I$ into $(n+1)$-simplices



triangles
$[v_0, v_1, w_1]$
$[v_0, w_0, w_1]$

$n = 1$



$n = 2$

3 tetrahedrons
$v_0 v_1 v_2 w_2$
$v_0 v_1 w_1 w_2$
$v_0 w_0 w_1 w_2$

For general n

$[v_0, v_1, \dots v_n] = \Delta^n \times \{0\}$ $\quad$ $[w_0, w_1, \dots, w_n] = \Delta^n \times \{1\}$

We divide $\Delta^n \times I$ into $(n+1)$-simplices

$$[v_0, v_1, \dots v_i, w_i, w_{i+1}, \dots w_n] \quad i = 0, 1, \dots n$$

n-simplex $[v_0, v_1, \dots v_i, w_{i+1}, \dots w_n]$ is the graph of the function $\varphi_i(t_0, \dots, t_n) = t_{i+1} + t_{i+2} + \dots + t_n$

$$0 = \varphi_n \leq \varphi_{n-1} \leq \dots \varphi_0 \leq \varphi_{-1} = 1.$$

The set between two graphs is $(n+1)$-simplex

$$[v_0, v_1 \dots v_i, w_i, w_{i+1} \dots w_n]$$

Consider a homotopy $F : X \times I \to Y$ between f and g, $F(-,0) = f$, $F(-,1) = g$. We will define chain homotopy

$$P : C_n(X) \to C_{n+1}(Y)$$

by the following formula $\quad \sigma : \Delta^n \to X \quad \sigma \in C_n(X)$

$$\boxed{P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times id_I) \Big/ [v_0, v_1 \dots v_i, w_i, w_{i+1} \dots w_n]}$$

$$\Delta^{n+1} \overset{p_0}{\underset{p_n}{\overset{\longrightarrow}{\underset{\vdots}{\longrightarrow}}}} \Delta^n \times I \xrightarrow{G\times id_I} X \times I \xrightarrow{F} Y$$

We show that $\partial P = g_* - f_* - P\partial$.

Geometric meaning: The chain given by the boundary of P on all $(n+1)$-simplices is the chain given by the boundary of P on the prism $\Delta^n \times I$, g_* is the boundary on the top of $\Delta^n \times I$, f_* is the boundary on the bottom and $P\partial$ is the boundary on the sides of the prism.

Now we confirm the idea above by a computation.

First compute $P(\partial\sigma) = P\left(\sum_{k=0}^{n} (-1)^k \sigma/[v_0 \dots \hat{v}_k \dots v_n]\right) =$

$$= \sum_{l<k} (-1)^k (-1)^l \, F\circ(G\times id_I)/[v_0 \dots v_l w_l \dots \hat{w}_k \dots w_n]$$

$$+ \sum_{l>k} (-1)^k (-1)^{l-1} \, F\circ(G\times id_I)/[v_0 \dots \hat{v}_k \dots v_l w_l w_{l+1} \dots w_n]$$

Now $$\partial P(\sigma) = \sum_{j\le i} (-1)^i (-1)^j \, F\circ(G\times id_I)/[v_0 \dots \hat{v}_j \dots v_i w_i \dots w_n]$$

$$+ \sum_{j\ge i} (-1)^i (-1)^{j+1} \, F\circ(G\times id_I)/[v_0 \dots v_i w_i \dots \hat{w}_j \dots w_n]$$

Now the summands for $j=i$ are cancelled pairwise with the exception of the first one in the first sum (it is $F\circ(G\times id)/[w_0 \dots w_n] = g(\sigma)$) and the last one in the second sum (it is $-F\circ(G\times id)/[v_0 \dots v_n] = -f(\sigma)$) and the rest it is $-P(\partial\sigma)$. So we have proved

$$\partial P(\sigma) = g(\sigma) - f(\sigma) - P(\partial\sigma). \qquad \square$$

Outline of the proof of Excision Theorem

Crucial lemma (without proof - see Hatcher)

Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of a space X. Let $C_n^{\mathcal{U}}(X)$ be free group generated by singular simpleces

$$\sigma : \Delta^n \longrightarrow X$$

which have the image $\sigma(\Delta^n)$ in some $U_\alpha \in \mathcal{U}$. Then the inclusion

$$C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$$

induces an isomorphism in homology (it is chain homology equivalence).

The same holds for pairs (X, A).

Proof of Excision Theorem (version B)

Take $\mathcal{U} = \{A, B\}$, $X = \operatorname{int} A \cup \operatorname{int} B$

now

$$0 \longrightarrow C_n(A) \longrightarrow C_n^{\mathcal{U}}(A \cup B) \longrightarrow \frac{C_n^{\mathcal{U}}(A \cup B)}{C_n(A)} \longrightarrow 0$$

and

$$\frac{C_n(B)}{C_n(A \cap B)} \xrightarrow[\text{iso}]{\cong} \frac{C_n^{\mathcal{U}}(A \cup B)}{C_n(A)} \xrightarrow[\text{equivalence}]{\text{chain. homology}} \frac{C_n(X)}{C_n(A)}$$

On homology it gives

$$H_n(B, A \cap B) \underset{\cong}{\longrightarrow} H_n^{\mathcal{U}}(X, A) \underset{\cong}{\longrightarrow} H_n(X, A) \qquad \square$$

Retraction and homology Let $A \hookrightarrow X$ and let $r : X \to A$ be a retraction. Then

$$H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \qquad r_* \circ i_* = id$$

$$H_n(X) \xrightarrow{r_*} H_n(A) \qquad \Downarrow$$

i_* is a mono

So we get

$$\begin{tikzcd} H_{n+1}(X,A) \arrow[r, "0"] & H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[l, bend left, "r_*"] \arrow[r, "j_*"] & H_n(X,A) \arrow[r, "0"] & H_{n-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

The short exact sequence splits and hence

$$H_n(X) \cong H_n(A) \oplus H_n(X,A)$$

<u>Reduced homology groups</u> For a space X with a basepoint x_0 we define <u>reduced homology groups</u> as

$$\overline{H}_n(X) := H_n(X, x_0)$$

Since $x_0 \hookrightarrow X$ has a retract $X \longrightarrow x_0$ we have

$$H_n(X) := H_n(x_0) \oplus H_n(X, x_0) = H_n(x_0) + \overline{H}_n(X)$$

So

$$\overline{H}_n(X) = H_n(X) \quad \text{for } n \geq 1$$

$$\overline{H}_0(X) \oplus \mathbb{Z} \cong H_0(X) \quad \text{for } n = 0$$

In particular $\overline{H}_n(\text{point}) \cong 0$ for all n

For pairs we define $\overline{H}_n(X,A) = H_n(X,A)$ if $A \neq \emptyset$.

<u>Lemma</u> If the horizontal sequences are exact, h is an isomorphism and the diagram commutes:

$$\begin{tikzcd} \arrow[r] & K_n \arrow[r, "i"] \arrow[d, "f"'] & L_n \arrow[r, "j"] \arrow[d, "g"] & M_n \arrow[r, "m"] \arrow[d, "h"] & K_{n-1} \arrow[r, "i"] \arrow[d, "f"] & L_{n-1} \arrow[r, "j"] \arrow[d, "g"] & M_{n-1} \arrow[r] \arrow[d, "h"] & {} \\ \arrow[r] & \overline{K}_n \arrow[r, "\bar{i}"'] & \overline{L}_n \arrow[r, "\bar{j}"'] & \overline{M}_n \arrow[r, "\bar{m}"'] & \overline{K}_{n-1} \arrow[r, "\bar{i}"'] & \overline{L}_{n-1} \arrow[r, "\bar{j}"'] & \overline{M}_{n-1} \arrow[r] & {} \end{tikzcd}$$

then the sequence

$$\longrightarrow K_n \xrightarrow{(f,i)} \overline{K}_n \oplus L_n \xrightarrow{\bar{i}-g} \overline{L}_n \xrightarrow{i \circ h^{-1} \circ \bar{j}} K_{n-1} \longrightarrow \overline{K}_{n-1} \oplus L_{n-1} \longrightarrow$$

is exact.

Proof : exercise

Application :

Mayer-Vietoris Theorem Let A and B are open in $X = A \cup B$. Then the sequence

$$H_n(A \cap B) \xrightarrow{i_A, i_B} H_n(A) \oplus H_n(B) \xrightarrow{j_A - j_B} H_n(A \cup B) \to H_{n-1}(A \cap B)$$

where $i_A : A \cap B \hookrightarrow A$, $i_B : A \cap B \hookrightarrow B$

$j_A : A \hookrightarrow X$, $j_B : B \hookrightarrow X$

is exact.

Proof Take long exact sequences for pairs $(B, B \cap A)$ and (X, A)

$$\begin{tikzcd}
H_n(A \cap B) \arrow[r, "i_B"] \arrow[d, "i_A"'] & H_n(B) \arrow[r] \arrow[d, "j_B"] & H_n(B, A \cap B) \arrow[r, "\partial_*"] \arrow[d, "\text{excision}", "\cong"'] & H_{n-1}(A \cap B) \arrow[d] \\
H_n(A) \arrow[r, "j_A"'] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r, "\partial_*"] & H_{n-1}(A)
\end{tikzcd}$$

The previous lemma gives the long exact sequence

$$\to H_n(A \cap B) \xrightarrow{i_A, i_B} H_n(A) \oplus H_n(B) \xrightarrow{j_A - j_B} H_n(X) \to H_{n-1}(A \cap B) \to$$

which is the Mayer-Vietoris exact sequence. $\square$

Remark We can get the same also for reduced homology groups!

$$\tilde{H}_n(A \cap B) \to \tilde{H}_n(A) \oplus \tilde{H}_n(B) \to \tilde{H}_n(A \cup B) \to H_{n-1}(A)$$

The long exact sequence of a triple (X, A, C)

Let $C \subseteq A \subseteq X$. We have the long exact sequence

$$H_n(A,C) \xrightarrow{i_*} H_n(X,C) \xrightarrow{j_*} H_n(X,A) \xrightarrow{D_*} H_{n-1}(A,C)$$

where $D_* : H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \longrightarrow H_{n-1}(A,C)$

Proof The sequence follows from the short exact sequence

$$0 \longrightarrow \frac{C_n(A)}{C_n(C)} \longrightarrow \frac{C_n(X)}{C_n(C)} \longrightarrow \frac{C_n(X)}{C_n(A)} \longrightarrow 0 .$$

$\square$

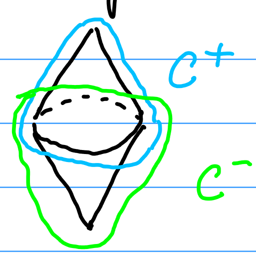
In particular it gives

$$\overline{H}_n(A) \longrightarrow \overline{H}_n(X) \longrightarrow \overline{H}_n(X,A) \longrightarrow \overline{H}_{n-1}(A)$$

Computation of homology groups of spheres

Computation of $\overline{H}_*(SX)$ using $\overline{H}_*(X)$

$SX = C^+X \cup C^-X$



$SX = C^+ \cup C^-$ open in SX

$\overline{H}_*(C^{\pm}X) \cong \overline{H}_*(pt) = 0$

$C^+ \cap C^- = X \times \{-\varepsilon, \varepsilon\} \simeq X$

M.-V. exact sequence gives

$$\underset{\underset{0}{\|}}{\overline{H}_n(C^+) \oplus \overline{H}_n(C^-)} \to H_n(SX) \xrightarrow[\cong]{\partial_*} \overline{H}_{n-1}(X) \to \underset{\underset{0}{\|}}{\overline{H}_{n-1}(C^+) \oplus \overline{H}_{n-1}(C^-)}$$

Especially: $\overline{H}_i(S^n) \cong \overline{H}_{i-1}(S^{n-1}) \simeq \dots \simeq \overline{H}_{i-n}(S^0)$

$H_0(S^0) \simeq H_0(\text{point}) \oplus H_0(\text{point}) \cong \mathbb{Z} \oplus \mathbb{Z}$ so

$\overline{H}_0(S^0) \cong \mathbb{Z}$, and $\overline{H}_j(S^0) = 0$ for $j \neq 0$.

It results in:

$$\overline{H}_i(S^n) = \begin{cases} \mathbb{Z} & \text{for } i = n \\ 0 & \text{for } i \neq n \end{cases}$$

Remark: $H_i(S^n)$ - unreduced

<u>Application</u> - degree of a mapping $f: S^n \to S^n$

Every map $f: S^n \to S^n$ induces in reduced homology $f_*: \overline{H}_n(S^n) \to \overline{H}_n(S^n)$

$$f_*: \mathbb{Z} \longrightarrow \mathbb{Z}$$

We define $\deg f \in \mathbb{Z}$ such that for any element $a \in \overline{H}_n(S^n)$ we have

$$f_*(a) = \deg f \cdot a$$

<u>Properties of degree</u>

1. $\deg id = 1$, $f \sim g \Rightarrow \deg f = \deg g$
2. $\deg(f \circ g) = \deg f \cdot \deg g$
3. $\deg(Sf) = \deg f$ $\qquad f: S^n \to S^n$, $Sf: SS^n \to SS^n$
4. $\deg\{(x_0, \dots x_n) \longmapsto (x_0, \dots, x_{i-1}, -x_i, x_{i+1}, \dots x_n)\} = -1$
5. $\deg(-id) = (-1)^{n+1}$
6. If $f: S^n \to S^n$ is not onto, then $\deg f = 0$.
7. If $f: S^n \to S^n$ has no fixed point, then $\deg f = (-1)^{n+1}$

<u>Proof</u>: 1), 2) clear.

3)

$$\begin{array}{ccc} \overline{H}_{n+1}(S^{n+1}) & \xrightarrow[\text{M.V. long ex. sequence}]{\cong} & \overline{H}_n(S^n) \\ \downarrow Sf_* & & \downarrow f_* \\ \overline{H}_{n+1}(S^{n+1}) & \xrightarrow{\cong} & \overline{H}_n(S^n) \end{array}$$

(4) $f : S^0 \to S^0 \quad f(-1) = 1,\ f(1) = -1$
generator of $\widetilde{H}_0(S^0)$ is given by the cycle $1 - (-1)$
and it maps to $(-1) - (1)$. The degree is -1.
If $f : S^n \to S^n$, $f(x_0, \dots x_n) = (x_0, \dots x_{i-1}, -x_i, \dots x_n)$
then $Sf : S^{n+1} \to S^{n+1}$ is $Sf(x_0 \dots x_i \dots x_{n+1})$
$= (x_0, \dots x_{i-1}, -x_i, \dots x_{n+1})$
and $\deg Sf = \deg f$

(5) $-id : S^n \to S^n$ is the composition of $(n+1)$ maps changing the sign of one coordinate. Hence
$$\deg(-id) = (-1)^{n+1}.$$

(6) If $f : S^n \to S^n$ is not onto, then f factors as

$$S^n \to S^n \setminus \{point\} \hookrightarrow S^n$$

$$\begin{array}{ccccc} \widetilde{H}_n(S^n) & \longrightarrow & \widetilde{H}_n(S^n \setminus point) & \longrightarrow & H_n(S^n) \\ \| & & \| & & \| \\ \mathbb{Z} & & 0 \ (\text{contractible}) & & \mathbb{Z} \end{array}$$

Hence the degree has to be 0.

(7) $f : S^n \to S^n$ without a fixed point is homotopic to $-id : S^n \to S^n$ via the homotopy

$$H(x,t) = \frac{t f(x) - (1-t)x}{\| t f(x) - (1-t) x \|} \qquad \begin{array}{ll} t=0 & -id \\ t=1 & f \end{array}$$

The homotopy is well defined:
$$t f(x) - (1-t)x \neq 0 \iff t f(x) \neq (1-t) x \Leftarrow f(x) \neq x$$
for $x, f(x) \in S^n$.

$\square$