# Lecture 5 : Homology of CW-complexes

Recall the definition of degree:

$f : S^n \to S^n \qquad \tilde{H}_n(f) : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n) : \tilde{H}_n(f)(a) = \deg f \cdot a$

**Local degree**: Let $f : S^n \to S^n$, $y \in S^n$ and $f^{-1}(y) = \{x_1, x_2, \dots x_m\}$, $U_i$ a neighbourhood of $x_i$, $U_i \cap U_j \neq \emptyset$ for $i \neq j$, $V$ a neighbourhood of $y$ and $f(U_i) \subseteq V$. Now

$$(f/U_i)_* : H_n(U_i, U_i \setminus \{x_i\}) \longrightarrow H_n(V, V-y)$$

Excision thm gives $H_n(U_i, U_i \setminus \{x_i\}) \cong H_n(S^n, S^n \setminus \{x_i\}) \cong \mathbb{Z}$
The second iso follows from the long exact sequence of the pair $(S^n, S^n \setminus \{x_i\})$. Similarly,

$$H_n(V, V \setminus \{y\}) \cong H_n(S^n, S^n \setminus \{y\}) \cong \mathbb{Z}.$$

Local degree

$$\deg f/x_i \in \mathbb{Z}$$

is the number such that $(f/U_i)_*(x) = \deg f/x_i \cdot x$

**Theorem**: Let $f : S^n \to S^n$, $f^{-1}(y) = \{x_1, x_2, \dots x_m\}$
Then

$$\deg f = \sum_{i=1}^{m} \deg f/x_i$$

Proof: In the notation above consider the diagram

$$\begin{tikzcd}
 & \bigoplus H_n(U_i, U_i \setminus x_i) \arrow[r, "\Sigma (f/U_i)_*"] \arrow[d, "k_1, \dots, k_m"] \arrow[dl, "\cong"'] & H_n(V, V \setminus y) \arrow[d, "\cong"] \\
\bigoplus H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_1 \dots p_m"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
 & H_n(S^n) \arrow[u, "j"] \arrow[r, "f_*"] \arrow[ul] & H_n(S^n) \arrow[u, "\cong"]
\end{tikzcd}$$

where $k_1 \sqcup k_2 \sqcup \dots \sqcup k_m : \sqcup (U_i, U_i \setminus x_i) \longrightarrow (S^n, S^n \setminus f^{-1}(y))$
and $p_i \circ k_i$ is an isomorphism
So $H_n(S^n, S^n \setminus f^{-1}(y)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ (m times)
and $j(1) = (1, 1, \dots, 1)$ $\quad j : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$
Now in the diagram

$$\begin{tikzcd}
(1\ 1\ 1 \dots 1) \arrow[r] \arrow[d, "k_1 \dots k_m"] & \sum_{i=1}^{m} \deg(f/U_i) \arrow[d] \\
(1\ 1\ 1 \dots 1) \arrow[r] & \bullet \\
1 \arrow[r] \arrow[u, "j"] & \deg f \arrow[u]
\end{tikzcd}$$

Hence $\deg f = \sum_{i=1}^{m} \deg(f/U_i)$. $\square$

<u>Application</u>: For every $n \geq 1$ and every $k \in \mathbb{Z}$ there is a map $f : S^n \to S^n$ such that
$$\deg f = k.$$

<u>Proof</u>: Consider $S^1 \subset \mathbb{C}$ and $f(z) = z^k : S^1 \to S^1$.

Then $f^{-1}(1) = \{ e^{i \frac{2\pi}{k}} \in S^1,\ i = 0, 1, \dots k-1 \}$ and $\deg(f/X_i) = 1$ for $k \geq 1$. Hence $\deg f = k$.
The degree of the constant map $f(z) = z^0 = 1$ is $0$.
The degree of $f(z) = z^{-k}$ is

$$\deg f = \deg(z^k) \cdot \deg\left(\frac{1}{z}\right) = k(-1) = -k.$$

Now we use the fact proved in the previous lecture that $\deg Sf = \deg f$.

# HOMOLOGY OF CW-COMLEXES

$X$ .. a CW-complex, $X^n$ its $n$-skeleton, $X^{-1} = \emptyset$ and $X^0/X^{-1} = X^0/\emptyset = X^0 \sqcup \{*\}$

Lemma: If $A \hookrightarrow X$ is a cofibration, then the projection

$$(X, A) \longrightarrow (X/A, *)$$

induces an isomorphism

$$H_n(X, A) \xrightarrow[\cong]{} \tilde{H}_n(X/A)$$

Proof: We have already shown that

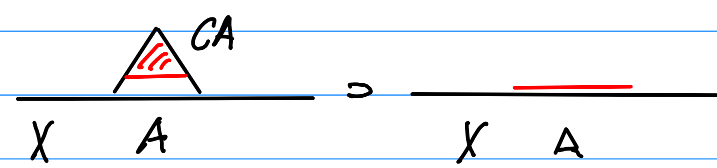
$$X \cup CA \sim X \cup CA/CA \cong X/A$$

Then

$$\tilde{H}_n(X \cup CA) \cong \tilde{H}_n(X \cup CA/CA) \cong H^n(X/A).$$

From the long exact sequence of the pair $(X \cup CA, CA)$ we get

$$\tilde{H}_n(X \cup CA) \xrightarrow{\cong} H_n(X \cup CA, CA) \xleftarrow[\cong]{\text{excision}} H_n(X, A)$$

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For every CW-complex $X$ we define a chain complex

$(C_*^{CW}, d_*)$ by $C_n^{CW} = H_n(X^n, X^{n-1})$ and by

$$\underbrace{H_n(X^n, X^{n-1}) \xrightarrow{\partial_*} H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X^{n-1}, X^{n-2})}_{d_n}$$

We know that

$$H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1}) \cong \tilde{H}_k\left(\bigvee_\alpha S^n_\alpha\right) \cong \begin{cases} \bigoplus_\alpha \mathbb{Z} & k=n \\ 0 & k \neq n \end{cases}$$

where $\alpha$ goes through all $n$-cells in $X$.

## Properties of $(C_*^{CW}(X), d_*)$

1. It is really a chain complex : $d \circ d = 0$
2. $H_k(X^n) = 0$ for $k \geq n+1$
3. $H_k(X^n) = H_k(X)$ for $k \leq n-1$
4. $$H_*(C_*^{CW}, d_*) \cong H_*(X)$$
   Homology groups of this complex are the same as the singular homology groups of $X$.

Proof: Consider the diagram

$$\begin{tikzcd}
 & 0 & & \\
 & H_n(X^{n+1}) \arrow[u] & & \\
0 \arrow[r] & H_n(X^n) \arrow[u, "i_n"] \arrow[dr, "j_n"] & & \\
 & H_{n+1}(X^{n+1}, X^n) \arrow[u, "\partial_*"] \arrow[r, "d_{n+1}"'] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"'] \arrow[dr, "\partial_*"'] & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & & H_{n-1}(X^{n-1}) \arrow[u, "j_{n-1}"'] \\
 & & & H_{n-1}(X^{n-2}) = 0 \arrow[u]
\end{tikzcd}$$

$d_n \circ d_{n+1} = 0$ follows from the diagram.

For $k \geq n+1$ we get

$$0 \cong H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}) \xrightarrow{\cong} H_k(X^n) \longrightarrow H_k(X^n, X^{n-1}) \cong 0$$

Using induction with respect to $n$ we get

$$H_k(X^n) \cong 0 .$$

For $k \leq n-1$ we get

$$0 \cong H_{k+1}(X^{n+1}, X^n) \to H_k(X^n) \xrightarrow{\cong} H_k(X^{n+1}) \to H_k(X^{n+1}, X^n) \cong 0$$

So we get $H_k(X^k) \cong H_k(X^{k+1}) \cong H_k(X^{k+2}) \cong \dots$

$$\cong \lim_{n\to\infty} H_k(X^n) \cong H_k(X) .$$

To prove the last statement let us return to the diagram:

$$\ker d_n = \ker \partial_* = \operatorname{im} j_n \cong H_n(X^n)$$

$\operatorname{im} d_{n+1} \cong \operatorname{im} \partial_*$ . Hence

$$\frac{\ker d_n}{\operatorname{im} d_{n+1}} \cong \frac{H_n(X^n)}{\operatorname{im} \partial_*} \cong \frac{H_n(X^n)}{\ker i_n} \cong H_n(X^{n+1}) \cong H_n(X).$$

$\blacksquare$

## Computation of $d_n$

$H_n(X^n, X^{n-1}) \cong \bigoplus_\alpha \mathbb{Z}_\alpha \qquad \alpha$ $n$-cell in $X$

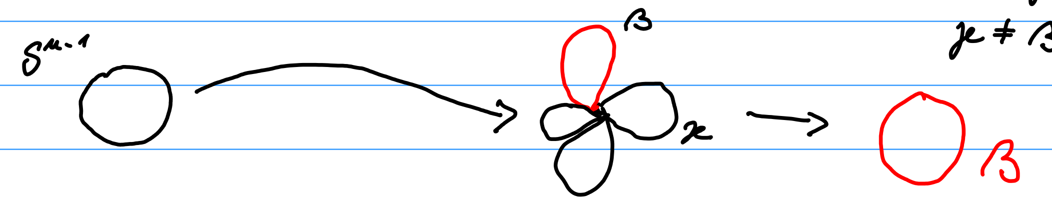
$H_{n-1}(X^{n-1}, X^{n-2}) \cong \bigoplus_\beta \mathbb{Z}_\beta \qquad \beta$ $(n-1)$-cell in $X$

So $\quad d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta}\, e_\beta^{n-1}$ ,

where

$d_{\alpha\beta}$ is the degree of the following map

$$S^{n-1} = \partial D^n_\alpha \xrightarrow{f_\alpha} X^{n-1} \longrightarrow X^{n-1}/X^{n-2} \longrightarrow X^{n-1}/X^{n-2} \cup \bigcup_{x \neq \beta} e^{n-1}_x$$




## (1) Homology of $\mathbb{C}P^n$

$\mathbb{C}P^n = e^0 \cup e^2 \cup \ldots \cup e^{2n}$, hence

$$C_i^{CW}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

$$0 \to 0 \to \overset{2n}{\mathbb{Z}} \to 0 \to \overset{2n-2}{\mathbb{Z}} \to \ldots \quad \overset{2}{\mathbb{Z}} \to \overset{1}{0} \to \overset{2}{\mathbb{Z}} \to 0 \ldots$$

The homology of this complex is

$$H_i(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

## (2) Homology of $\mathbb{R}P^n$

$\mathbb{R}P^n = e^0 \cup e^1 \cup e^2 \cup \ldots \cup e^n$, so

$$C_i^{CW}(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

We want to compute the differential

$$d(e^n) = \deg g \; e^{n-1}$$

where $g: \quad S^{n-1} \xrightarrow[\text{attaching map}]{} X^{n-1} \longrightarrow X^{n-1}/X^{n-2} \cong S^{n-1}$

$$g: \quad S^{n-1} \longrightarrow \mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} \cong S^{n-1}$$

Every point in $S^{n-1}$ has to preimages $x_1$ and $x_2 = -x_1$ in $S^{n-1}$.

$g$ on the neighbourhood of $x_i$ is a homeomorphism, hence $\deg g/x_i = \pm 1$.

Next $\quad g/U_2 = g/U_1 \circ (-id)$

That is why

$$\deg g/x_2 = \deg g/x_1 \cdot (-1)^n$$

Consequently

$$\deg g = \pm 1 (1 + (-1)^n) = \begin{matrix} 0 & \text{for } n \text{ odd} \\ \pm 2 & \text{for } n \text{ even} \end{matrix}$$

$C_*^{CW}(\mathbb{R}P^n)$

$$\begin{matrix} \ldots & \mathbb{Z} & \xrightarrow{2\times} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2\times} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow 0 \\ & 4 & & 3 & & 2 & & 1 & & 0 & \end{matrix}$$

So we obtain

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } k=0 \text{ and } k=n \text{ odd} \\ \mathbb{Z}/2 & \text{for } k \text{ odd}, \ 0 < k < n \\ 0 & \text{in other cases} \end{cases}$$

# Homology groups with coefficients

Let $G$ be an Abelian group. We define a chain complex $(C_*(X;G), \partial_*^G)$ as

$$C_n(X;G) = C_n(X) \otimes G \quad , \quad \partial_n^G = \partial_n \otimes id_G$$

and homology group of $X$ with coefficients in $G$ as

$$H_n(X;G) = H_n(C_*(X;G), \partial_*^G).$$

These functors have the same properties as $H_n$, only exception is

$$H_n(*;G) = \begin{cases} G & \text{for } n=0 \\ 0 & \text{otherwise} \end{cases}$$

We have that $H_n(X) = H_n(X;\mathbb{Z})$.

Generally it is not true that

$$H_n(X;G) = H_n(X) \otimes G$$

Computation of $H_*(X;G)$ for $X$ CW-complex is very similar to what we have done

$$C_n^{CW}(X;G) = H_n(X^n, X^{n-1}) \otimes G, \; d_n^G = d_n \otimes id_G$$

Using this we compute $H_*(\mathbb{R}P;\mathbb{Z}/2)$

$C_*^{CW}(\mathbb{R}P^n)$

$$\to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} 0$$

Hence

$$H_i(\mathbb{RP}^n; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{for } 0 \le i \le n \\ 0 & \text{otherwise} \end{cases}$$