Lecture 6 : SINGULAR COHOMOLOGY

Cochain complex :
$$\cdots \to C^{n-1} \xrightarrow{\delta} C^n \xrightarrow{\delta} C^{n+1} \to \cdots \qquad \delta\circ\delta = 0$$
δ.. coboundary differential

Homomorphism $f^* : (C^*, \delta) \to (D^*, \delta)$

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[r, "\delta"] \arrow[d, "f^{n-1}"] & C^n \arrow[r, "\delta"] \arrow[d, "f^n"] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & D^{n-1} \arrow[r, "\delta"] & D^n \arrow[r, "\delta"] & D^{n+1} \arrow[r] &
\end{tikzcd}$$

Cochain homotopy between $f^*, g^* : C^* \to D^*$ are group homomorphisms $s^n : C^n \to D^{n-1}$ such that
$$\delta \circ s^n + s^{n+1} \circ \delta = f^n - g^n$$

Cohomology groups $\quad H^n(C^*, \delta) = \dfrac{\ker \delta^n}{\operatorname{im} \delta^{n-1}}$

Elements of C^* are called cochains.
Elements of $\ker \delta^n$ are called cocycles.
Elements of $\operatorname{im} \delta^{n-1}$ are called coboundaries.

Every homo of cochain complexes $f^* : C^* \to D^*$ induces a homo of cohomology groups
$$H^n(f^*) : H^n(C^*) \to H^n(D^*)$$

If f^* and g^* are cochain homotopic, then
$$H^n(f^*) = H^n(g^*).$$

Singular cohomology groups

X a topological space, $C_*(X)$ its singular chain complex, G an Abelian group. We define singular cochain complex with coefficients in G as

$$C^n(X;G) = \mathrm{Hom}(C_n(X), G)$$

with coboundary operator

$$\delta(f)(c) = f(\partial c)$$

where $\partial : C_{n+1}(X) \to C_n(X)$ is the boundary operator in C_*.

$$\begin{tikzcd} C_{n+1}(X) \arrow[r, "\partial"] \arrow[rr, bend right, "\partial f"'] & C_n(X) \arrow[r, "f"] & G \end{tikzcd}$$

Then singular cohomology groups of X are cohomology groups of this cochain complex. Notation

$$H^n(X;G) = H^n(C^*(X;G), \delta)$$

Singular cohomology of a pair (X,A)

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} \frac{C_n(X)}{C_n(A)} \to 0$$

Apply $\mathrm{Hom}(-, G)$:

$$(*) \quad 0 \leftarrow \mathrm{Hom}(C_n(A), G) \xleftarrow{i^*} \mathrm{Hom}(C_n(X), G) \xleftarrow{j^*} \underbrace{\mathrm{Hom}\left(\frac{C_n(X)}{C_n(A)}, G\right)}_{\cong \ker i^*} \leftarrow 0$$

Cohomology groups of the cochain complex

$$\mathrm{Hom}\left(\frac{C_n(X)}{C_n(A)}, G\right)$$

singular cohomology groups $H^n(X,A;G)$.

(*) is a short exact sequence from which we get a long exact sequence of singular cohomology groups

$$\cdots \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta^*} H^{n+1}(X,A;G) \to \cdots$$

where δ^* is a connecting homomorphism.

Singular cohomology is a contravariant functor from Top^2 into graded Abelian groups. It is the following composition:

$(X,A) \longmapsto C_*(X,A) \longmapsto Hom(C_*(X,A);G) \longmapsto H^*(X,A;G)$

So every $f:(X,A) \to (Y,B)$ induces a homomorphism

$$f^*: H^n(Y,B;G) \to H^n(X,A;G)$$

The properties of singular cohomology groups are analogous to properties of singular homology groups. One only "turns" arrows.

Excision Thm Let $A \subseteq X$ and $\overline{C} \subseteq \operatorname{int} A$. Then the inclusion $(X \setminus C, A \setminus C) \to (X,A)$ induces an isomorphism

$$H^n(X \setminus C, A \setminus C) \xleftarrow{\cong} H^n(X,A).$$

Homotopy invariance If $f,g : (X,A) \to (Y,B)$ are homotopic, then

$$H^n(f) = H^n(g) : H^n(Y,B) \to H^n(X,A).$$

Cohomology of a point

$$H^n(*;G) \cong \begin{cases} G & n=0 \\ 0 & \text{otherwise} \end{cases}$$

Reduced cohomology groups

$$\tilde{H}^n(X;G) = H^n(X, x_0; G)$$
$$\tilde{H}^n(X,A;G) = H^n(X,A;G)$$

Mayer-Vietoris exact sequence

Let $A, B \subseteq X$, $X = \operatorname{int} A \cup \operatorname{int} B$. Then there is a long exact sequence

$$\longrightarrow H^n(A \cup B) \longrightarrow H^n(A) \oplus H^n(B) \longrightarrow H^n(A \cap B) \longrightarrow H^{n+1}(A \cup B)$$

Also a version for reduced cohomology

Using this exact sequence we get

$$\tilde{H}^k(S^n;G) = \begin{cases} G & \text{for } k=n \\ 0 & \text{otherwise} \end{cases}$$

Computation of cohomology of CW-complexes

can be carried out from the cochain complex

$C^*_{CW}(X) = (\operatorname{Hom}(H_n(X^n, X^{n-1}), G), d^n)$

It enables us to compute $H^*(\mathbb{C}P^n;G)$ and $H^*(\mathbb{R}P^n;G)$.

Product in cohomology

Let R be a commutative ring with 1.

Cup product $\varphi \in C^k(X;R)$,
$\psi \in C^\ell(X;R)$, we define $\varphi \cup \psi \in C^{k+\ell}(X;R)$:

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|[v_0 \ldots v_k]) \cdot \psi(\sigma|[v_k, \ldots v_{k+\ell}])$$

Lemma:

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$$

The proof consists in a computation.

Lemma implies: Cup product of cocycles is a cocycle.
Cup product of a cocycle and a coboundary is a coboundary.
So we get induced products in cohomology:

$$\cup : H^k(X;R) \otimes H^\ell(X;R) \longrightarrow H^{k+\ell}(X;R)$$

$$\cup : H^k(X,A;R) \otimes H^\ell(X;R) \longrightarrow H^{k+\ell}(X,A;R)$$

$$\cup : H^k(X,A;R) \otimes H^\ell(X,A;R) \longrightarrow H^{k+\ell}(X,A;R)$$

$$\cup : H^k(X,A;R) \otimes H^\ell(X,B;R) \longrightarrow H^{k+\ell}(X, A\cup B;R)$$

for A and B open or subcomplexes.

Properties of the cup product

1) If $X \neq \emptyset$, $1 \in H^0(X;R)$ is a unit for $\cup$
2) $\cup$ is associative
3) $\cup$ is graded commutative
$$\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$$
4) $\cup$ is natural
$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

Cross product $\times : H^k(X) \otimes H^\ell(Y) \longrightarrow H^{k+\ell}(X\times Y)$

$$\begin{tikzcd} & X\times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

Let $\alpha \in H^k(X;R)$ and $\beta \in H^\ell(X;R)$. Then we define $\alpha\times\beta \in H^{k+\ell}(X\times Y;R)$:

$$\alpha\times\beta = p_1^*(\alpha) \cup p_2^*(\beta)$$

So we have defined cross product using cup product. We can also express the cup product from the cross product. Consider the diagonal map

$$\Delta : X \longrightarrow X\times X \qquad \Delta(x) = (x,x)$$

In cohomology $\Delta^* : H^*(X\times X) \longrightarrow H^*(X)$

If $\alpha\in H^k(X;R)$, $\beta \in H^\ell(X;R)$ then

$$\alpha\cup\beta = \Delta^*(\alpha\times\beta)$$

It follows from the fact that both compositions

$$X \xrightarrow{\Delta} X\times X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$$

are id_X. Hence

$$\Delta^*(\alpha\times\beta) = \Delta^*(p_1^*(\alpha)\cup p_2^*(\beta)) = \Delta^* p_1^*(\alpha)\cup\Delta^* p_2^*(\beta) = \alpha\cup\beta.$$

Some homological algebra - tensor product of chain and cochain complexes

$$(C_*\otimes D_*)_n = \bigoplus_{k=0}^{n} C_k\otimes D_{n-k}$$

$$\partial^{C\otimes D} = \partial^C \otimes id_D + (-1)^{deg}\, id_C \otimes \partial^D$$

Similarly for $C^* \otimes D^*$

If C^*, D^* have a product and δ is a graded derivation with respect to it, then
$C^* \otimes D^*$
has a product with the same property:

$$(a \otimes b) \cdot (c \otimes d) := (-1)^{|b||c|} (a \cdot_C c) \otimes (b \cdot_D d)$$

Theorem: Cross product is a homeomorphism of graded rings:

$$H^*(X) \otimes H^*(Y) \xrightarrow{\times} H^*(X \times Y).$$

Proof: Let us write ω for the cross product $\times$. We have to prove that

$$\omega((a\otimes b)\cdot(c \otimes d)) = \omega(a \otimes b) \cup \omega(c \otimes d)$$

Let us compute

$$\omega((a\otimes b)\cdot(c\otimes d)) = \omega\left((-1)^{|b||c|}(a \cup c) \otimes (b \cup d)\right) =$$

$$= (-1)^{|b|\,|c|}\, p_1^*(a \cup c) \cup p_2^*(b \cup d) = (-1)^{|b||c|}\, p_1^*(a) \cup p_1^*(c) \cup$$

$$\cup\, p_2^* b \cup p_2^* d = \left(p_1^*(a) \cup p_2^*(b)\right) \cup \left(p_1^*(c) \cup p_2^*(d)\right) =$$

$$= (a \times b) \cup (c \times d) = \omega(a \otimes b) \cup \omega(c \otimes d).$$ ∎

Computation of $H^*(X \times Y)$ using $H^*(X)$ and $H^*(Y)$ is called Künneth formula.

Theorem: Let (X, A), (Y, B) be pairs of CW-complexes and let $H^k(Y, B; R)$ be free finitely generated R-modules. Then

$$\mu : H^*(X, A) \otimes H^*(Y, B) \longrightarrow H^*(X \times Y; A \times Y \cup X \times B)$$

is an isomorphism of graded rings.

Examples: (1) $H^*(S^n \times S^k) \cong H^*(S^n) \otimes H^*(S^k)$
$$\cong \mathbb{Z}[\alpha, \beta] / (\alpha^2, \beta^2)$$

where $\alpha \in H^n(S^n)$ and $\beta \in H^k(S^k)$ are generators

More precisely:

$$H^*(S^n) \cong \mathbb{Z}[\alpha]/\alpha^2 \quad , \quad H^*(S^k) \cong \mathbb{Z}[\beta]/\beta^2$$

$$H^*(S^n) \otimes H^*(S^k) \cong \mathbb{Z}[\alpha \otimes 1, 1 \otimes \beta] / \alpha^2 \otimes 1, 1 \otimes \beta^2$$

(2) $H^*(X \times S^n; \mathbb{Z}) \cong H^*(X)[\alpha] / (\alpha^2)$ $\quad \alpha \in H^n$