Lecture 11 HOMOTOPY AND CW-COMPLEXES

Last time: notions of n-connectivity and n-equivalence

(X,A) $\pi_i(X,A)=0$ for $1\le i\le n$, $f: X\to Y$ $\pi_i(X)\to\pi_i(Y)$ iso $i\le n-1$, epi $i=n$

Compression lemma: (X,A) pair of CW-complexes, (Y,B) pair of spaces

$$\pi_n(Y,B,y_0)=0 \quad \text{for all } y_0\in B$$

whenever there is an n-cell in $X\setminus A$.

Then every $f:(X,A)\longrightarrow(Y,B)$ is homotopic rel A to a map $g: X\longrightarrow B$.

$$\begin{tikzcd}
A \arrow[r, "f/A"] \arrow[d, hook] & B \arrow[d, hook] \\
X \arrow[r, "f"'] \arrow[ur, "g", "\sim"'] & Y
\end{tikzcd}$$

Theorem Let $h: Z\to Y$ be an n-equivalence. For every CW-complex X the induced map

$$h_*: [X,Z]\longrightarrow[X,Y]$$

is

(1) surjection if $\dim X\le n$,

(2) bijection if $\dim X\le n-1$.

Proof: If h is an inclusion we use compression lemma. If $\dim X\le n$ for this situation

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & Z \arrow[d, hook, "h"] \\
X \arrow[r] \arrow[ur] & Y
\end{tikzcd}$$

It implies (1).

For $\dim X \leq n-1$, for this situation

$$\begin{tikzcd}
X\times\{0\}\cup X\times\{1\} \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[d, hook] & Z \arrow[d, hook, "h"] \\
X\times I \arrow[r, "f"'] \arrow[ur, "H"] & Y
\end{tikzcd}$$

which implies (2).

If h is not an inclusion, we replace Y by the cylinder of h:

$$\begin{tikzcd}
 & & Z \arrow[dl, "h"'] \arrow[d, "i_Z", "\simeq"'] \arrow[dr, "h"] & \\
X \arrow[r] \arrow[urr] & Y \arrow[r, "\simeq", "i_Y"'] & M_h \arrow[r, "\simeq", "p"'] & Y
\end{tikzcd}$$

Weak homotopy equivalence

A map $f: X \to Y$ is called a weak homotopy equivalence if

$$f_*\ \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

is an isomorphism for all $n \geq 0$ and $x_0 \in X$.

WHITEHEAD THEOREM

Let $h: Z \to Y$ be a weak equivalence between two CW-complexes. Then h is a homotopy equivalence.

Proof: If h is an inclusion we again apply the compression lemma:

$$\begin{tikzcd} Z \arrow[r, "id_Z"] \arrow[d, "h"'] & Z \arrow[d, "h"] \\ Y \arrow[r, "id_Y"'] \arrow[ur, "g"] & Y \end{tikzcd} \qquad \pi_n(Y,Z) = 0 \text{ for all } n$$

g is a homotopy inverse of h.

If h is not an inclusion we use the mapping cylinder of h.

SIMPLICIAL APPROXIMATION LEMMA

Assumptions: $f : I^n \longrightarrow Z = W \cup e^k$ where W is a space and e^k is a k-cell.

Conclusion: There is a map $f_1 : I^n \longrightarrow Z$ and a simplex $\Delta^k \subset e^k$ such that

(1) $f_1 \sim f$ rel $f^{-1}(W)$

(2) $f_1^{-1}(\Delta^k)$ is a union of finitely many convex polyhedra such that f_1 on the polyhedra is an affine projection (surjective) $\mathbb{R}^n \longrightarrow \mathbb{R}^k$. (If $k > n$, then $f_1^{-1}(\Delta^k)$ is empty.) $f_1 : (x_1, x_2, \dots x_n) \longmapsto (x_1, x_2, \dots x_k)$

Hatcher Lemma 4.40 (350-351)

Further we need a version with $k \leq n$.

CELLULAR APPROXIMATION THEOREM

Let $f: X \to Y$ is a map between two CW-complexes, which is cellular on subcomplex $A \subset X$. Then there is a cellular map $g: X \to Y$ such that

$$g \sim f \quad \text{rel } A.$$

Corollary 1: $\pi_k(S^n) = 0$ for $k < n$. $\{s_0\}$

$f: (S^k, s_0) \to (S^n, s_0)$ $\quad f_1: (S^k, s_0) \to (k\text{-skeleton of } S^n, s_0)$

Corollary 2: Let (X, A) be a pair of CW-complexes and let $X \setminus A$ contain cells of dim $> n$. Then the pair (X, A) is n-connected.

Proof of Cor 2 Every class in $\pi_k(X, A)$, $k \le n$ contains a cellular representative g $I^k \to X^k \subseteq A$. Hence $[g] = 0$ in $\pi_k(X, A)$. See the criterion after definition of hom. groups

Proof of cellular appr. thm: By induction

$f_{-1} = f$

$f_n: X \to Y$, f_n is cellular on X^n

$$f_n \sim f_{n-1} \quad \text{rel } X^{n-1} \cup A$$

If we have such a sequence of maps we can define

$$g(x) = f_n(x) \quad \text{for } x \in X^n$$

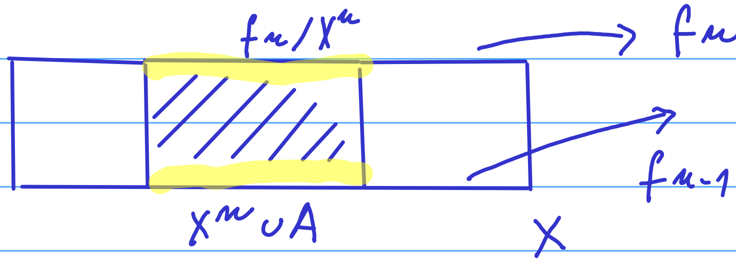
and we will have $g \sim f$ rel A.

Induction step: Suppose we have f_{n-1}. $f_{n-1}(e^n)$ does not lie in Y^n for an n-cell e^n. Then $f(e^n)$ has an intersection with a cell e^k in Y, $k > n$. According to simpl. appr. lemma, there is $h: \bar{e}^n \to Y$, $h \sim f_{n-1}/\bar{e}_n$ rel ∂e^n and there is a $\Delta^k \subset e^k$, $h(\bar{e}^n) \subset Y \setminus \Delta^k$.

∂e^k is a deformation retract of $\bar{e}^k \setminus \Delta^k$, hence there is $g : \bar{e}^n \to Y \setminus e^k$, $g \sim h$ rel ∂e^n. We repeat this procedure until we get a map $\bar{e}^n \to Y$ with image in Y^n, which is homotopic to $f_{n-1}/\bar{e}_n$ rel $\partial \bar{e}_n$.

So we get $f_n : X^n \to Y^n$ $\quad f_n \sim f_{n-1}$ rel $A \cup X^{n-1}$

Using HEP we extend f_n on the map $X \to Y$.



Approximation of spaces by CW-complexes

Let (X,A) be a pair, X a general space, A a CW-complex. A pair of CW-complexes (Z,A) is called

n-connected CW model for (X,A)

if there is a map $f : Z \to X$ such that
(1) $f/A = id_A$,
(2) $f_* \ \pi_i(Z) \to \pi_i(X)$ is an iso for $i > n$,
(3) $f_* \ \pi_n(Z) \to \pi_n(X)$ is a mono.

If A contains a point from every component of path connectivity of X, then 0-connected model $f : Z \to X$ is a weak homotopy equivalence.

CW approximation theorem

For every $n \in \mathbb{N}$ and every pair (X, A) where A is a CW-complex there is an n-connected CW-model
$$f : (Z, A) \longrightarrow (X, A)$$
such that $Z \setminus A$ has only cells of dim $> n$.

Proof by induction

$$A = Z_n \subset Z_{n+1} \subset \ldots\ldots Z_{k-1} \subset Z_k \subset \ldots \quad Z$$

Z_k arises from Z_{k-1} by attaching cells of dim k.

$$f : Z_k \to X, \quad f|A = id_A$$
$$f_* \quad \pi_i(Z_k) \to \pi_i(X) \quad \text{mono for } n \le i < k$$
$$\text{epi for } n < i \le k$$

Suppose X, A path connected, $x_0 \in A$ fixed.

We get Z_{k+1} from Z_k in two steps:

$$\begin{tikzcd} & & X & \\ Z_k \arrow[urr] & \subset Y_{k+1} \arrow[ur] & \subset Z_{k+1} \arrow[u] \end{tikzcd}$$

- We have $f : Z_k \to X$. Let $\varphi_\alpha : S^k \to Z_k$ which are generators of
$$\ker(f_* : \pi_k(Z_k) \longrightarrow \pi_k(X))$$
Put

$$Y_{k+1} = Z_k \cup_{\varphi_\alpha} \bigcup D_\alpha^{k+1}$$

$S^k \xrightarrow{\varphi_\alpha}$

$f : Z_k \to X$ can be extended to $f : Y_{k+1} \to X$ due to the fact that $f_*[\varphi_\alpha] = 0 \in \pi_k(X)$.

$$\pi_i(Y_{k+1}) = \pi_i(Z_k) \quad \text{for } i \le k-1$$

according to cellular appr. thm.

$$S^i \to Y_{k+1}$$

$$\pi_k(Z_k) \longrightarrow \pi_k(Y_{k+1}) \xrightarrow{epi} \pi_k(X) \qquad \text{epi}$$

is an epi according to assumption on

$$f_* \quad \pi_k(Z_k) \longrightarrow \pi_k(X).$$

Hence extended f gives also epi

$$f_* : \pi_k(Y_{k+1}) \longrightarrow \pi_k(X)$$

We prove that $f_* : \pi_k(Y_{k+1}) \longrightarrow \pi_k(X)$ is a mono.
Let $[\varphi] \in \pi_k(Y_{k+1})$ and $f \circ \varphi \sim 0$.
$\varphi : S^k \longrightarrow Y_{k+1}$ is homotopic to $\bar\varphi : S^k \longrightarrow Y^k_{k+1} = Z_k$
and $[f\bar\varphi] = 0$ in $\pi_k(X)$.
That is why $[\bar\varphi] \in \ker f_*$ and consequently

$$[\bar\varphi] = \sum [\varphi_\alpha].$$

Now $[\varphi_\alpha] = 0$ in $\pi_k(Y_{k+1})$, and so $[\bar\varphi] = 0$ in $\pi_k(Y_{k+1})$.

Conclusion $\qquad f_* : \pi_i(Y_{k+1}) \longrightarrow \pi_i(X)$ mono $n \le i \le k$
and epi $n < i \le k$.

- Let $\psi_\beta : S^{k+1}_\beta \longrightarrow X$ be generators of $\pi_{k+1}(X)$.

Put

$$Z_{k+1} = Y_{k+1} \vee \bigvee_\beta S^{k+1}_\beta$$

and define $f = \psi_\beta$ on S^{k+1}_β.

$id: S^{k+1} \to S^{k+1}_\beta \longmapsto \psi_\beta \; S^{k+1} \to X$

Then $f_* : \pi_{k+1}(Z_{k+1}) \longrightarrow \pi_{k+1}(X)$ is an epimorphism.
Next $\pi_i(Z_{k+1}) = \pi_i(Y_{k+1})$ for $i \le k$ and so

$$f_* : \pi_i(Z_{k+1}) \longrightarrow \pi_i(X)$$

are epi according to assumptions of Y_{k+1}.

Similarly $f_*: \pi_i(Z_{k+1}) \longrightarrow \pi_i(X)$
are mono for $i \leq k-1$ using cellular appr. theorem
and properties of f on Y_{k+1}.

It remains to show that
$$f_*: \pi_k(Z_{k+1}) \longrightarrow \pi_k(X)$$
is a mono.
We use the long exact sequence of the pair (Z_{k+1}, Y_{k+1})

$$\begin{tikzcd}
\pi_{k+1}(Z_{k+1}, Y_{k+1}) \arrow[r] & \pi_k(Y_{k+1}) \arrow[r, two heads, "\text{epi}"] \arrow[dr, "\cong"', "f_*"] & \pi_k(Z_{k+1}) \arrow[r, "0"] \arrow[d, "f_*"', "\text{mono}"] & \pi_k(Z_{k+1}, Y_{k+1}) = 0 \arrow[d, "0"] \\
 & \text{iso} & \pi_k(X) & \pi_{k-1}(Y_{k+1}) \arrow[d, "\cong"] \\
 & & & \pi_{k-1}(Z_{k+1})
\end{tikzcd}$$

<u>Corollary</u>: n-connected pair of CW-complexes (X, A) is homotopy equivalent to a pair of CW-complexes (Z, A) where $Z - A$ has only cells in dim $\geq n+1$.

$f: Z \longrightarrow X$ $\quad$ $f_*: \pi_i(Z) \longrightarrow \pi_i(X)$ $\quad i > n$ iso

$f_*: \pi_n(Z) \longrightarrow \pi_n(X)$ $\quad$ mono

$\pi_n(A) \xrightarrow[i]{\cancel{f_*}} \pi_n(X)$ $\quad$ epi

n-equivalence

$[\varphi] \in \pi_n(Z)$ $\quad \tilde{\varphi}: S^n \longrightarrow Z^{(n)} = A$

$f_*[\varphi] = i_*[\tilde{\varphi}]$ $\quad$ i_* is an epi

$\Rightarrow f_*$ is an epi

$i < n$ $\quad f_* = i_*$ $\quad i_*: \pi_n(A) \longrightarrow \pi_n(X)$ is an iso

iso $\downarrow$

$\pi_n(Z)$ - - -> also iso

f_* is a weak hom. equiv.

which because Z and X are CW-complexes means according to Whitehead thm that f is a homotopy equivalence.