# Lecture 12: Homotopy excision and Hurewicz Theorem

## Homotopy excision (BLAKERS-MASSEY THEOREM)

Let $A, B$ are subcomplexes in the CW-complex $X = A \cup B$.
Let $C = A \cap B$ be connected. If the pair $(A, C)$ is $m$-connected & the pair $(B, C)$ is $n$-connected, then the inclusion

$$(A, C) \hookrightarrow (\overset{A\cup B}{\overset{\shortparallel}{X}}, B)$$

is an $(n+m)$-equivalence.

Compare with excision theorem for homology groups!
For the proof see the text to the lecture (Chapter 13) or Hatcher.

**Corollary:** Let $(X, A)$ be an $r$-connected pair of CW-complexes & let $A$ be $s$-connected. Then the map

$$X \to X/A$$

is an $(r+s+1)$-equivalence.

*Proof:* The pair $(X, A)$ is $r$-connected according to the assumption and the pair $(CA, A)$ is $(s+1)$-connected because

$$\pi_{i+1}(CA, A) \cong \pi_i(A)$$

from the long exact sequence of the pair $(CA, A)$.
The Blakers-Massey Theorem gives that

$$(X, A) \longrightarrow (X \cup CA, CA)$$

is $(r+s+1)$-equivalence. Further

$$\pi_i(X \cup CA, CA) \longleftarrow \pi_i(X \cup CA)$$

is an isomorphism (since $CA$ is contractible) and

$$X \cup CA \longrightarrow X \cup CA/CA$$

is a homotopy equivalence (since $CA$ is contractible in itself) and

$$X \cup CA/CA \longleftarrow X/A$$

is a homeomorphism

$$\begin{tikzcd}
(X,A) \arrow[r, hook, "(r+s+1)\text{-equiv}"] \arrow[drr, bend right, "(r+s+1)\text{-equiv}"'] & (X \cup CA, CA) \arrow[r, "\infty\text{-equiv}"] & X \cup CA/CA \arrow[d, "\cong \quad \infty\text{-equiv}"] \\
 & X \cup CA \arrow[u, "\infty\text{-equiv}"] \arrow[ur, "\infty\text{-equiv}"'] & X/A
\end{tikzcd}$$

## Freudenthal Theorem

Let $X$ be $(n-1)$-connected CW-complex, $n \geq 1$. Then the suspension homomorphism

$$S : \pi_i(X) \longrightarrow \pi_{i+1}(SX)$$
$$f \longmapsto Sf$$

is an isomorphism for $i \leq 2n-2$ and an epimorphism for $i = 2n-1$.

**Proof:** $SX = C_+X \cup C_-X$, $C_+X \cap C_-X = X$

The pairs $(C_+X, X)$ an $(C_-X, X)$ are $n$-connected

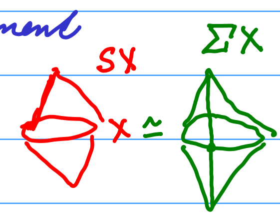
We get

$$\begin{tikzcd}
\pi_i(X) & \pi_{i+1}(C_+X, X) \arrow[l, "\partial"', "\cong"] \arrow[r, "\cong \text{ for } i+1 \leq 2n-1,\ \text{epi for } i+1 = 2n"] & \pi_{i+1}(SX, C_-X) \arrow[d, "\cong \text{ previous statement}"] \\
 & & \pi_{i+1}(SX/C_-X) \\
 & & \pi_{i+1}(SX) \arrow[u, "\cong"']
\end{tikzcd}$$

$C_+f,\ f$ LONG EXACT SEQ. $\ni Sf,\ C_-f$

$f \xrightarrow{S} \ \ni Sf$

We have to show that it is really suspension.



## Stable homotopy groups

The Freudenthal Theorem holds not only for CW-complexes but also for all topological spaces. The proof is based on the fact that for every top. space $Z$ we can find a CW-complex $X$ and a weak homotopy equivalence $X \xrightarrow{f} Z$. Using the diagram

$$\begin{array}{ccc} \pi_i(X) & \xrightarrow[\cong]{f_*} & \pi_i(Z) \\ S_X \downarrow \cong & & \downarrow S_Z \;\; \cong \\ \pi_{i+1}(SX) & \xrightarrow[\cong]{Sf_*} & \pi_{i+1}(SZ) \end{array}$$

we get that $S_Z$ is an iso if $S_X$ is an iso and that $S_Z$ is an epi if $S_X$ is an epi.

If $X$ is $n$-connected, then $SX$ is $(n+1)$-connected. So in the sequence of suspension maps

$$\pi_i(X) \longrightarrow \pi_{i+1}(SX) \longrightarrow \pi_{i+2}(S^2X) \longrightarrow \dots \pi_{i+n}(S^nX) \to \dots$$

we get from a certain point isomorphisms.

If $X$ is not connected, then $SX$ is connected, $S^2X$ is 1-connected, etc, $S^nX$ is $(n-1)$-connected and so

$$\pi_i(S^nX) \longrightarrow \pi_{i+1}(S^{n+1}X)$$

is an iso for $i \le 2n-2$. For such $i$

$$\pi_{i+j}(S^{n+j}X) \longrightarrow \pi_{i+j+1}(S^{n+j+1}X)$$

are iso for all $j \ge 0$, because

$$i+j \le 2n+2j-2.$$

If for fixed $i$ we take $n \ge i+2$, we get

$$i+n \le 2n-2$$

and hence $\pi_i(X) \to \dots \pi_{i+n}(S^nX) \xrightarrow[\cong]{} \pi_{i+n+1}(S^{n+1}X)$

is an isomorphism.

We define stable homotopy groups as

$$\pi_i^s(X) = \operatorname*{colim}_{n \to \infty} \pi_{i+n}(S^n X).$$

**Theorem** The group $\pi_n(S^n)$ is isomorphic to $\mathbb{Z}$ with generator given by the identity

$$id : S^n \to S^n.$$

Moreover, the isomorphism

$$\pi_n(S^n) \xrightarrow[\cong]{} \mathbb{Z} \qquad [f] \mapsto \deg f$$

is given by the degree of maps.

**Proof:**

$$\begin{tikzcd}
& \pi_1(S^1) \arrow[r, "epi"] \arrow[d, "deg"', "\cong"] & \pi_2(S^2) \ni f \arrow[r, "iso"', "S"] \arrow[d, "\cong"', "deg"] & \pi_3(S^3) \ni Sf \arrow[r, "iso"] \arrow[d, "\cong"', "deg"] & \cdots \\
S^1 \cong \mathbb{R}/\mathbb{Z} & \mathbb{Z} \arrow[r, "id"] & \mathbb{Z} \ni \deg f \arrow[r, "id"] & \mathbb{Z} \ni \deg Sf \arrow[r] & \cdots
\end{tikzcd}$$

Since $\deg Sf = \deg f$, we get subsequently, that $\deg\ \pi_i(S^i) \xrightarrow[\deg]{} \mathbb{Z}$ are isomorphisms.

**Lemma**

$$\pi_n\Big(\prod_{\alpha \in A} X_\alpha\Big) \cong \prod_{\alpha \in A} \pi_n(X_\alpha)$$

$f\ S^n \to \prod X_\alpha \qquad f = (f_\alpha) : S^n \to X_\alpha$

**Lemma** For $n \geq 2$

$$\pi_n\Big(\bigvee_{\alpha \in A} S^n_\alpha\Big) = \bigoplus_{\alpha \in A} \mathbb{Z}$$

**Proof:** (1) If $A$ is finite, then $\bigvee_A S^n_\alpha$ is a sub-complex in $\prod_{\alpha \in A} S^n_\alpha$. The pair $(\prod S^n_\alpha, \bigvee S^n_\alpha)$

is $(2n-1)$-connected since all the cells in $\prod S^n_\alpha \setminus \vee S^n_\alpha$ are of dimension $2n$ and higher.
That is why for $n \geq 2$

$f: S^n \to \prod S^n_\alpha$ $\quad \pi_n(\vee S^n_\alpha) \cong \pi_n(\prod S^n_\alpha) \cong \prod \pi_n(S^n_\alpha) \cong \oplus \pi_n(S^n_\alpha)$

since the product of finite number of abelian groups is the same as their sum.

(2) $A$ infinite. Then

$$\Phi : \oplus \pi_n(S^n_\alpha) \longrightarrow \pi_n(\vee S^n_\alpha)$$

is induced by maps $S^n_\alpha \longrightarrow \vee S^n_\alpha$. $\Phi$ is an isomorphism since every map into $\vee S^n_\alpha$ from $S^n$ or $S^n \times I$ goes only into finite number of spheres.

**Lemma** Let $n \geq 2$ and let

$$X = \bigvee_{\alpha \in A} S^n_\alpha \cup_{\varphi_\beta} \cup e^{n+1}_\beta$$

where

$$\varphi_\beta : S^n_\beta \longrightarrow \vee S^n_\alpha$$

is an attaching map for $e^{n+1}_\beta$. Then

$f: S^i \to X$
$g: S^i \to X^i = *$

$$\pi_i(X) = \begin{cases} 0 & \text{for } i \leq n-1 \\ \bigoplus_{\alpha \in A} \pi_n(S^n_\alpha) / N & i = n \end{cases}$$

where $N$ is a subgroup generated by $[\varphi_\beta]$.

$[\varphi_\beta] \in \pi_n(\vee S^n_\alpha) = \oplus \pi_n(S^n_\alpha) \mid \varphi_\beta : S^n_\beta \to \vee S^n_\alpha$

Proof: The long exact sequence for the pair $(X, X^n = \bigvee_{\alpha \in A} S^n_\alpha)$
gives

$$\pi_{n+1}(X, X^n) \xrightarrow{\partial} \pi_n(X^n) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, X^n) = 0.$$

The pair $(X, X^n)$ is $n$-connected, $X^n$ is $(n-1)$-connected, hence

$$\pi_{n+1}(X, X^n) \xrightarrow[\cong]{} \pi_{n+1}(X/X^n) = \pi_{n+1}(\vee S^{n+1}_\beta) = \bigoplus_{\beta\in B} \mathbb{Z}$$

That is why

$$\pi_n(X) \cong \pi_n(X^n)/\underline{\operatorname{Im}\partial} \cong \bigoplus_{\alpha\in A}\mathbb{Z} \,/\, \underline{N}$$

We show that $\operatorname{Im}\partial \cong N$.

$$\begin{tikzcd}
\pi_{n+1}(X, \vee_\alpha S^n_\alpha) \arrow[r, "\partial"] \arrow[d, "\cong"] & \pi_n(\vee S^n_\alpha) \\
\pi_{n+1}(X/\vee_\alpha S^n_\alpha) = \pi_{n+1}(\vee_\beta S^{n+1}_\beta) &
\end{tikzcd}$$

Generators in the group $\pi_{n+1}(X, \vee_\alpha S^n_\alpha)$ are maps

$$\begin{tikzcd}
D^{n+1}_\beta \arrow[r, "\Phi_\beta"] & X \\
\partial D^{n+1}_\beta \arrow[u, hook] \arrow[r, "\varphi_\beta"'] & \vee S^n_\alpha \arrow[u, hook]
\end{tikzcd}$$

and $\partial[\Phi_\beta] = [\varphi_\beta]$. Hence $\operatorname{Im}\partial = N =$ the group generated by $[\varphi_\beta]$.

## Hurewicz homomorphism

For every space $X$ we define a map

$$h : \pi_n(X) \longrightarrow H_n(X)$$

as $\qquad h[f] = f_*(s) \in H_n(X)$

where $f : S^n \longrightarrow X$ and $s \in H_n(S^n) \cong \mathbb{Z}$ is a generator.

$$f_* \; H_n(\underset{s}{S^n}) \longrightarrow H_n(X)$$

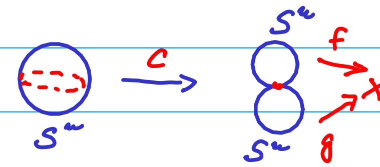
Similarly $\qquad h : \pi_n(X, A) \longrightarrow H_n(X, A)$

$$h[f] = f_*(s)$$

where $f : (D^n, S^{n-1}) \longrightarrow (X, A)$
and $s \in H_n(D^n, S^{n-1}) \cong \mathbb{Z}$
is a generator.

**Lemma:** $h : \pi_n(X) \longrightarrow H_n(X)$
is a group homomorphism.

**Proof:** Consider
$$c : S^n \longrightarrow S^n \vee S^n$$




$$\begin{tikzcd} H_n(S^n) \arrow[r, "c_*"] & H_n(S^n \vee S^n) \arrow[r, "(f\vee g)_*"] \arrow[d, shift left, "q_{1*} \oplus q_{2*}"] & H_n(X) \\ & H_n(S^n) \oplus H_n(S^n) \arrow[u, shift left, "i_{1*} + i_{2*}"] & \end{tikzcd}$$

$$\begin{tikzcd} S^n \arrow[r] \arrow[rr, bend right, "f+g"'] & S^n \vee S^n \arrow[r, "f\vee g"] & X \end{tikzcd}$$

$$h([f]+[g]) = (f+g)_*(s) = (f \vee g)_* \, c_*(s)$$

$$= (f \vee g)_* (i_{1*} + i_{2*}) (q_{1*} \oplus q_{2*}) \, c_*(s) =$$

$$= (f \vee g)_* (i_{1*} + i_{2*}) (s \oplus s) = f_*(s) + g_*(s) = h[f] + h[g]$$

Homomorphism $h : \pi_n(X) \longrightarrow H_n(X)$, resp.
$h : \pi_n(X, A) \longrightarrow H_n(X, A)$, is called
HUREWICZ HOMOMORPHISM.

Hurewicz homomorphism is natural: For
$f : (X, A) \longrightarrow (Y, B)$ we have commutative
diagram

$$\begin{tikzcd} \pi_n(X, A) \arrow[r, "h"] \arrow[d, "f_*"'] & H_n(X, A) \arrow[d, "f_*"] \\ \pi_n(Y, B) \arrow[r, "h"] & H_n(Y, B) \end{tikzcd}$$

and moreover

$$\begin{tikzcd}
\pi_{n+1}(X,A) \arrow[r, "h"] \arrow[d, "\partial"] & H_{n+1}(X,A) \arrow[d, "\partial"] \\
\pi_n(A) \arrow[r, "h"] & H_n(A)
\end{tikzcd}$$

We will prove it in the tutorial.

## HUREWICZ THEOREM

Let $n \geq 2$. If $X$ is $(n-1)$-connected, then $\tilde{H}_i(X) = 0$ for $i \leq n-1$ and

$$h : \pi_n(X) \longrightarrow H_n(X)$$

is an isomorphism.

Proof: Let $X$ be a $CW$-complex and

$$X = \bigvee_{\alpha \in A} S^n_\alpha \cup_{\varphi_\beta} \bigcup e^{n+1}_\beta .$$

Then

$$\tilde{H}_i(X) = 0 \quad \text{for } i \leq n-1 .$$

Further

$$\begin{tikzcd}
& \oplus\, \pi_n(S^n_\alpha) & & & \\[-20pt]
\pi_{n+1}(X, X^n) \arrow[r, "\partial"] \arrow[d, "\cong", "h"'] & \pi_n(X^n) \arrow[r] \arrow[d, "\cong", "h"'] & \pi_n(X) \arrow[r] \arrow[d, "h"] & 0 \arrow[r] \arrow[d, "\cong", "h"'] & 0 \arrow[d, "\cong"] \\
H_{n+1}(X, X^n) \arrow[r, "\partial"] & H_n(X^n) \arrow[r] & H_n(X) \arrow[r] & 0 \arrow[r] & 0 \\[-20pt]
& \oplus\, H_n(S^n_\alpha) & & &
\end{tikzcd} \qquad (*)$$

$$\begin{tikzcd}
\pi_{n+1}(X, X^n) \arrow[r, phantom, "\cong"] \arrow[d, "h"] & \pi_{n+1}(X/X^n) \arrow[r, phantom, "\cong"] \arrow[d, "h"] & \oplus\, \pi_{n+1}(S^{n+1}_\beta) \arrow[d, "h"] \\
H_{n+1}(X, X^n) \arrow[r, phantom, "\cong"] & H_{n+1}(X/X^n) \arrow[r, phantom, "\cong"] & \oplus\, H_{n+1}(S^{n+1}_\beta)
\end{tikzcd}$$

$h : \pi_{n+1}(S^{n+1}) \xrightarrow{iso} H_{n+1}(S$

is an iso

Since in the diagram $(*)$ the first and second homomorphisms are iso, the third one is also iso.

For general $X$ we know that $\pi_n(X) = \pi_n(X^{n+1})$

and $H_n(X) = H_n(X^{n+1})$. So the proof is completed for any CW-complex $X$.

$$
\begin{array}{ccc}
\pi_n(X^{n+1}) & \xrightarrow[\cong]{h} & H_n(X^{n+1}) \\
\downarrow \cong & & \downarrow \cong \\
\pi_n(X) & \xrightarrow[\cong]{h} & H_n(X)
\end{array}
$$