Exercise 1 Show $\partial\partial = 0$ in singular chain complexes.

Use and prove the formula $\varepsilon^i_{n+1} \circ \varepsilon^j_n = \varepsilon^{j+1}_{n+1} \circ \varepsilon^i_n$ for $i \leq j$.

$C_n(X)$ ... free ab. group on generators sing n-simpl. $\quad \sigma : \Delta^n \longrightarrow X$

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \varepsilon^i_n \qquad \varepsilon^i_n : \Delta^{n-1} \longrightarrow \Delta^n \quad (t_0, \dots t_n) \quad \sum_{i=0}^{n} t_i = 1$$

$$- \quad \varepsilon^i_n (t_0, t_1, \dots t_{n-1}) = (t_0, \dots t_{i-1}, 0, t_i, \dots t_{n-1})$$

$$i < j$$

$$L = \varepsilon^i_{n+1} \circ \varepsilon^j_n (t_0 \dots t_{n-1}) = \varepsilon^i_{n+1} (t_0 \, t_i \, t_{j-1}, 0 \, t_j \dots t_{n-1})$$

$$= (t_0, \dots t_{i-1}, 0, t_i, \dots t_{j-1}, 0, t_j, \dots t_{n-1})$$

$$P = \varepsilon^{j+1}_{n+1} \circ \varepsilon^i_n (t_0 \dots t_{n-1}) = \varepsilon^{j+1}_{n+1} (t_0, \dots t_{i-1}, 0, t_i, \dots t_{n-1})$$

$$= (t_0, \dots t_{i-1}, 0, t_i, \dots t_{j-1}, 0, t_j, \dots t_{n-1})$$

$$L = P$$

$\sigma \in C_{n+1}(X)$ $\quad \sigma$ is $(n+1)$-sing. simplex, $\quad \partial c \in C_n(X)$, $\quad \partial\partial c \in C_{n-1}(X)$

$$\partial(\partial c) = \partial\left(\sum_{i=0}^{n+1} (-1)^i \sigma \circ \varepsilon^i_{n+1}\right) = \sum_{i=0}^{n+1} (-1)^i \left(\sum_{j=0}^{n} (-1)^j (\sigma \circ \varepsilon^i_{n+1}) \circ \varepsilon^j_n\right)$$

$$= \sum_{1 \leq i \leq j \leq n} (-1)^{i+j} \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^j_n + \sum_{1 \leq j < i \leq n+1} (-1)^{i+j} \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^j_n$$

Here we rename the indices $i \leftrightarrow j$

$$= \sum_{1 \leq i \leq j \leq n} (-1)^{i+j} \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^j_n + \sum_{i < j} (-1)^{i+j} \sigma \circ \varepsilon^j_{n+1} \circ \varepsilon^i_n$$

Next we will write $j+1$ instead of j!

$$= \sum_{i \leq j} (-1)^{i+j} \, \sigma \circ \underline{\varepsilon_{n+1}^{i} \circ \varepsilon_n^{j}} + \sum_{i \leq j} (-1)^{i+j+1} \, \sigma \circ \underline{\varepsilon_{n+1}^{j+1} \circ \varepsilon_n^{i}}$$

Underlined expressions are the same but with opposite signs in the sum. That is why the sum is zero!

Exercise 2 Compute simplicial homology of $\partial\Delta^2$ (boundary of a triangle).

Simplicial complex

combinatorially set of points S ordered

set of simplices $\mathcal{T}$

$\sigma \in \mathcal{T}$ $\sigma \subseteq S$ ordered

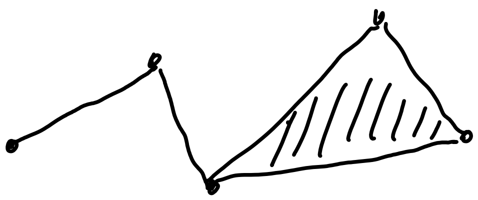
$\sigma_1 < \sigma_2 \wedge \sigma_2 \in \mathcal{T} \Rightarrow \sigma_1 \in \mathcal{T}$

C $\forall s \in S$ $\{s\} \in \mathcal{T}$

points, segments, triangles, tetrahedrons, n-simplices in $\mathcal{T}$

Geometric realization take S in $\mathbb{R}^\infty$ such that all points are affinely independent

Geometric realization is union of geometric simplices from $\mathcal{T}$



We can define simplicial homology of a simplicial complex

$$\partial[v_0 v_1 \dots v_n] = \sum (-1)^i [v_0 \dots \hat{v}_i \dots v_n]$$

$C_n(X)$ .... is the free ab. group generated by n-simplices from $\mathcal{T}$

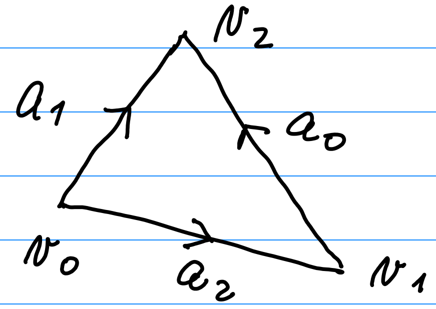
Often this group is finitely gen

$$C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} C_{n-2}(X)$$

Simplicial homology is

$$H_n(X) = \frac{\ker \partial_n}{\mathrm{im}\, \partial_{n+1}}$$

Compute simplicial homology for $\partial \Delta^2$.



$T = \{(v_0)(v_1)(v_2), a_0, a_1, a_2\}$

$$C_0(X) = \mathbb{Z}[v_0, v_1, v_2]$$

$$C_1(X) = \mathbb{Z}[a_0, a_1, a_2]$$

$$C_n(X) = 0 \qquad n \neq 0, 1$$

$$\partial v_i = 0 \qquad \underset{[v_1, v_2]}{\partial a_0} = v_2 - v_1 \qquad \partial a_1 = v_2 - v_0$$

$$\partial a_2 = v_1 - v_0 \qquad \partial : C_1 \longrightarrow C_0$$

$\ker \partial$, $\mathrm{im}\, \partial$

$$\begin{array}{cc} & \begin{array}{cc} \text{edges} & \mathrm{im}\,\partial \end{array} \\ \begin{array}{c} a_0 \to \\ a_1 \to \\ a_2 \to \end{array} & \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \end{array} \sim \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 \end{array}\right)$$

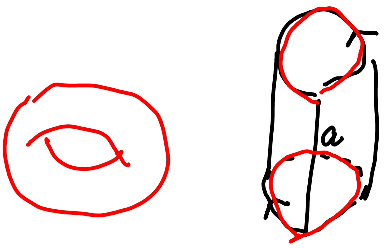
$$\sim \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} \partial(a_0 - a_1 + a_2) = 0 \\ \mathrm{im}\, \partial = \mathbb{Z}[-v_0 + v_2, -v_1 + v_2] \end{array}$$

$$H_1(X) = \frac{\ker \partial_1}{\mathrm{im}\, \partial_2} = \ker \partial_1 = \mathbb{Z}[a_0 - a_1 + a_2] \cong \mathbb{Z}$$

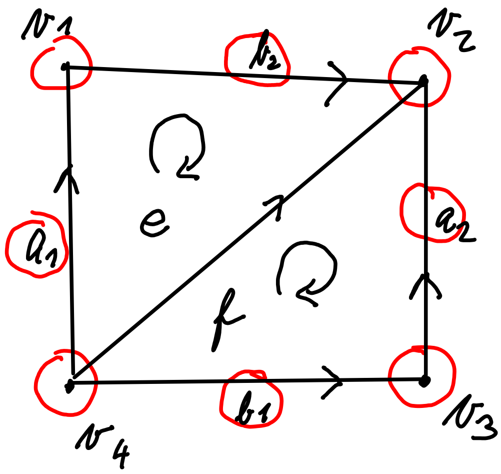
$$H_0(X) = \frac{\ker \partial_0}{\mathrm{im}\, \partial_1} = \frac{\mathbb{Z}[v_0, v_1, v_2]}{\mathbb{Z}[-v_0 + v_2, -v_1 + v_2]} = \frac{\mathbb{Z}[-v_0 + v_2, -v_1 + v_2, v_2]}{\mathbb{Z}[-v_0 + v_2, -v_1 + v_2]}$$

$$H_0(X) = \mathbb{Z}[v_0] \cong \mathbb{Z}$$

Exercise 3 Compute simplicial homology of the torus

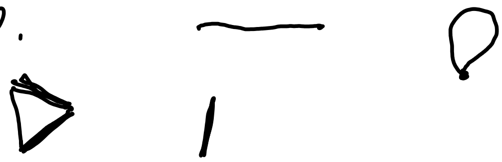


Torus as Δ-complex.



Torus as Δ-complex

We can generalize simplicial complexes to so called Δ-complexes.



$C_0(T) = \mathbb{Z}[v]$

chain complex

$C_1(T) = \mathbb{Z}[a, b, c]$

$C_2(T) = \mathbb{Z}[e, f]$

$\partial_0 v = 0 \quad \partial_1 a = v - v = 0, \quad \partial_1 b = 0, \quad \partial_1 c = 0$

$\partial_2 e = a + b - c \qquad \partial_2 f = c - a - b$

$\ker \partial_2 = e + f$

$\operatorname{im} \partial_2 = \mathbb{Z}[a+b-c]$

$\ker \partial_1 = \mathbb{Z}[a, b, c] \qquad H_2(T) = \dfrac{\ker \partial_2}{\operatorname{im} \partial_3} =$, $\operatorname{im} \partial_3 = 0$

$\operatorname{im} \partial_1 = 0$

$\ker \partial_0 = \mathbb{Z}[v] \qquad = \mathbb{Z}[e+f] \cong \mathbb{Z}$

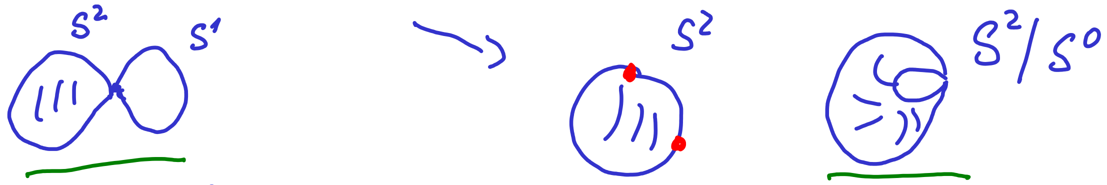
$H_0(T) = \dfrac{\ker \partial_0}{\operatorname{im} \partial_1} = \dfrac{\mathbb{Z}}{0} = \mathbb{Z}$

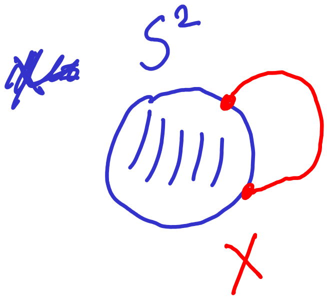
$H_1(T) = \dfrac{\ker \partial_1}{\operatorname{im} \partial_2} = \dfrac{\mathbb{Z}[a, b, c]}{\mathbb{Z}[a+b-c]} = \dfrac{\mathbb{Z}[a, b, a+b-c]}{\mathbb{Z}[a+b-c]}$

$\cong \mathbb{Z}[a, b] \cong \mathbb{Z} \oplus \mathbb{Z}$

Exercise 4 Prove that $S^2 \vee S^1$ is homotopy equivalent to S^2/S^0. ($S^2 \vee S^1 \simeq S^2/S^0$) Use the criterion:

Let (X, A) be a pair with HEP and A is contractible in itself. Then projection $X \to X/A$ is a homotopy equivalence.



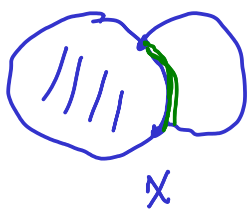


$A =$

A is contractible in itself

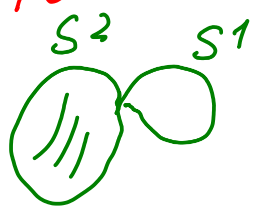
$X \simeq X/A$

S^2/S^0



$A =$

$X \simeq X/A$

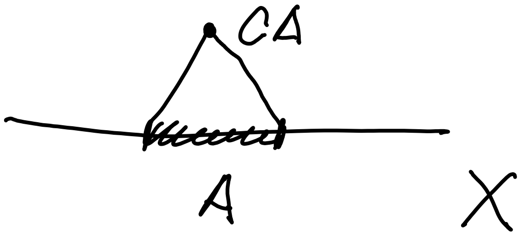


Exercise 5 Let $i : A \hookrightarrow X$ be a cofibration. Show that $X/A \simeq X \cup CA$ (cone of the map i).

(X, A) has HEP, A is a subcomplex in CW-complex X

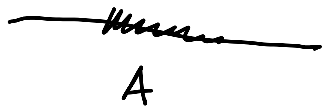
$$X/A \simeq X \cup CA = C_i$$

$$CA = A \times I / A \times \{1\}$$



$$CA \hookrightarrow X \cup CA$$

CA is contractible in itself



$$X \cup CA \underset{\text{hom. equiv.}}{\simeq} X \cup CA / CA \underset{\text{homeo}}{\cong} X/A$$

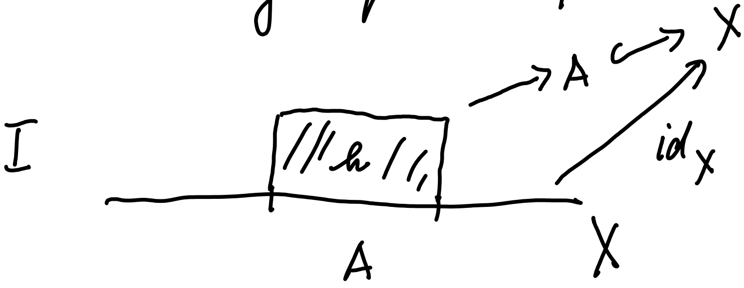
Exercise 6 Prove the criterion of hom. equivalence.

(X,A) has HEP, A is contractible in itself

$q: X \to X/A$ is a homotopy equivalence

We need „hom. inverse" $g: X/A \to X$

$$g \circ q \sim id_X \qquad q \circ g \sim id_{X/A}$$



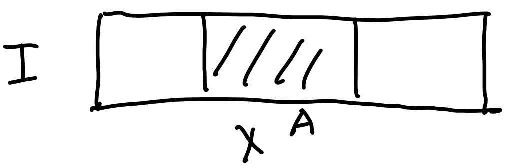
$h: A \times I \to A$

$h(-,0) = id_A$

$h(-,1) = $ point

$h \cup id_X$

HEP $\Rightarrow$ existence of f



f extends $h \cup id_X$

$$\begin{tikzcd}
X \times I \arrow[r, "f"] \arrow[d, "q \times id_I"'] & X \arrow[d, "q"] \\
X/A \times I \arrow[r, "\bar{f}"] & X/A \arrow[u, bend right, dashed, "g"']
\end{tikzcd}$$

$$\bar{f}([x],0) = [f(x,0)] = [x] \qquad id_{X/A}$$

$$\bar{f}([x],t) = [f(x,t)] = q(f(x,t))$$

$x \in A \Rightarrow f(x,t) \in A$

$$g: X/A \to X \qquad g([x]) = f(x,1)$$

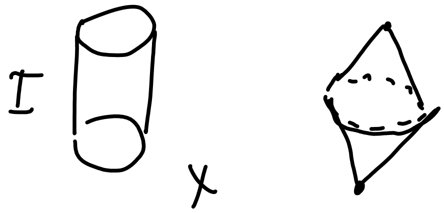
$$g \circ q(x) = g[x] = f(x,1) \sim id_X \text{ via } f$$

$$q \circ g([x]) = q(f(x,1)) = [f(x,1)] = \bar{f}([x],1)$$

$$\sim id_{X/A} \text{ via } \bar{f}$$

Homotopy equivalence of reduced and unreduced suspensions for CW-complexes.

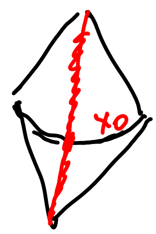
Unreduced suspension $SX = X \times I \Big/ \begin{matrix} (x_1,0)\sim(x_2,0) \\ (x_1,1)\sim(x_2,1) \end{matrix}$



SX

Reduced suspension $x_0 \in X$, a base point

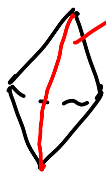
$$\Sigma X = SX / x_0 \times I$$



If X is a CW-complex, then

$$\Sigma X \simeq SX$$

$A = x_0 \times I$ contractible in itself

SX  $\simeq SX / x_0 \times I = \Sigma X$

X is a CW-complex, x_0 is zero cell

(X, x_0) has HEP

$(SX, x_0 \times I)$ is the pair of CW-complex and a subcomplex

$\Rightarrow$ has HEP so we can use the criterion above

SX  X has CW-structure $\Rightarrow$ SX has also CW-structure

Exercise 8 Given the comm. diagram with exact horizontal sequences

$$\begin{tikzcd}
\cdots \arrow[r] & K_n \arrow[r, "f"] \arrow[d, "k"] & L_n \arrow[r, "g"] \arrow[d, "\ell"] & M_n \arrow[r, "h"] \arrow[d, "m", "\cong"'] & K_{n-1} \arrow[r, "f"] \arrow[d, "k"] & L_{n-1} \arrow[r, "g"] \arrow[d, "\ell"] & M_{n-1} \arrow[r] \arrow[d, "m", "\cong"'] & \cdots \\
\cdots \arrow[r] & \overline{K_n} \arrow[r, "\overline{f}"'] & \overline{L_n} \arrow[r, "\overline{g}"'] & \overline{M_n} \arrow[r, "\overline{h}"'] & \overline{K_{n-1}} \arrow[r, "\overline{f}"'] & \overline{L_{n-1}} \arrow[r, "\overline{g}"'] & \overline{M_{n-1}} \arrow[r] & \cdots
\end{tikzcd}$$

there is a long exact sequence

$$\to K_n \xrightarrow{f \oplus k} L_n \oplus \overline{K_n} \xrightarrow{\overline{f} - \ell} \overline{L_n} \xrightarrow{h \circ m^{-1} \circ \overline{g}} K_{n-1} \xrightarrow{f \oplus k}$$

It is easy to prove that

$$\mathrm{im} \subseteq \ker$$

in all three cases. Next time we prove opposite inclusions.