Exercise 1. *Compare $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$, when distinguished points (are / are not) path connected.*

$f \cdot w$

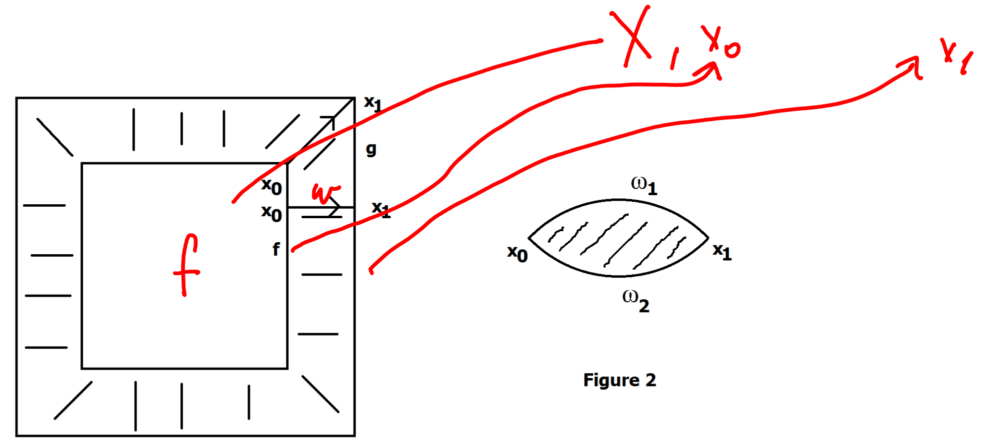


Figure 1

Figure 2

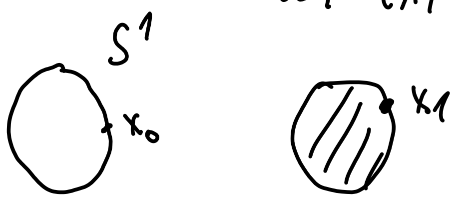
$X \qquad x_0, x_1 \in X$

There is no path between x_0 and x_1 in X, we cannot say anything about relation between $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$

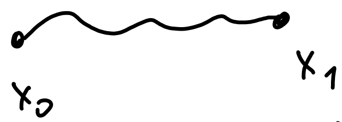
$X = S^1 \sqcup D^2 \qquad x_0 \in S^1, \quad x_1 \in D^2$

$$\pi_1(X, x_0) = \pi_1(S^1, x_0) \cong \mathbb{Z}$$

$$\pi_1(X, x_1) = \pi_1(D^2, x_1) \cong 0$$



There is a path w from x_0 to x_1



we will show that w induces isomorphism

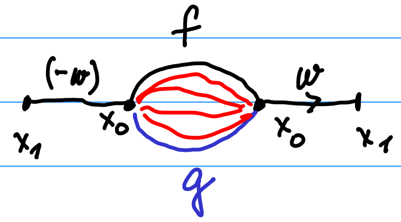
$$\pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$$

$$f : (I^n, \partial I^n) \longrightarrow (X, x_0) \rightsquigarrow (I^n, \partial I^n) \longrightarrow (X, x_1)$$

$$f \longmapsto f \cdot w$$

(1) $f \sim g \Rightarrow f\cdot w \sim g \cdot w$

$n=1$



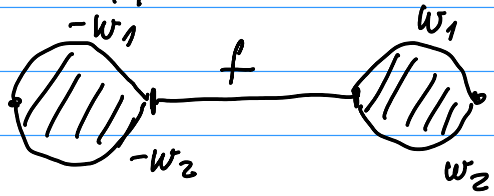
$f\cdot w$

$g\cdot w$

$f\cdot w \sim g\cdot w$

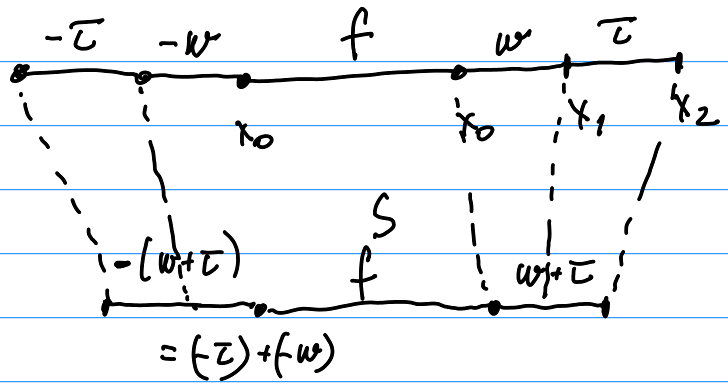
(2) $w_1 \sim w_2$ $\qquad f\cdot w_1 \sim f\cdot w_2$

$x_0 \rightsquigarrow x_1$



(3) $w \quad x_0 \longrightarrow x_1 \qquad \tau \quad x_1 \longrightarrow x_2 \qquad w+\tau : x_0 \longrightarrow x_2$

$f\cdot(w+\tau) \sim f\cdot w + f\cdot\tau$



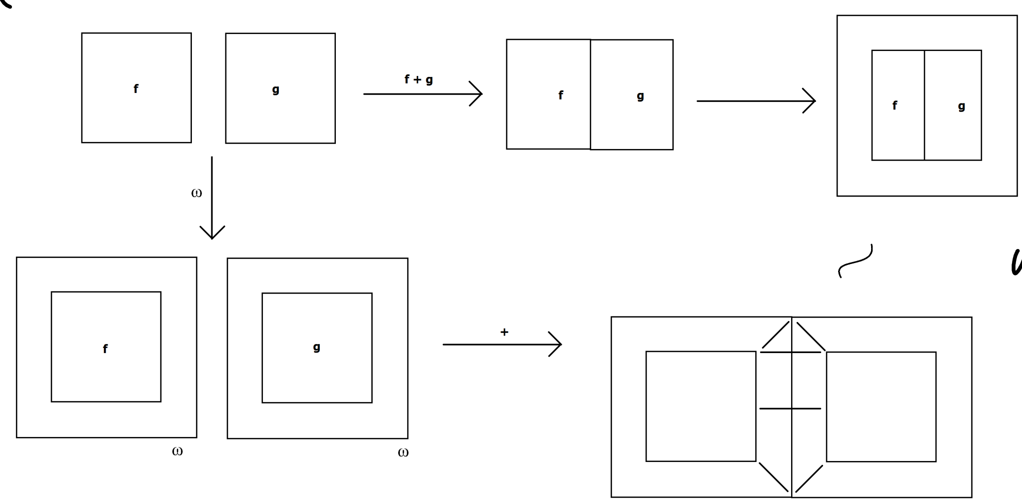
Corollary $\qquad (f\cdot w)(-w) \sim f(w+(-w)) \sim$

$f\cdot \text{const} \sim f$

$[f] \longmapsto [f\cdot w] \qquad \pi_n(X,x_0) \leftrightarrow \pi_n(X,x_1)$

$[g\cdot(-w)] \longleftarrow [g] \qquad \pi_n(X,x_0) \longleftarrow \pi_n(X,x_1)$

$(f+g)\cdot\omega \sim f\cdot\omega + g\cdot\omega$



$(f+g)\cdot\omega$

$\omega+(-\omega) \sim$ const

$f\cdot\omega+g\cdot\omega$

Figure 3

$[f] \longmapsto [f\cdot\omega] \quad \pi_n(X,x_0)\to\pi_n(X,x_1)$

is an isomorphism.

Exercise 2. *Using Van Kampen Theorem compute the fundamental group of the sphere with k holes.*

$\pi_1(X, x_0)$ using $\pi_1(U)$, $\pi_1(V)$

$X = U \cup V$

$X, U, V, U \cap V$ path connected

$x_0 \in U \cap V$

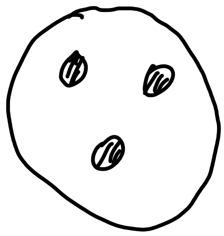
$$\pi_1(X, x_0) = \frac{\pi_1(U) * \pi_1(V)}{N}$$

N normal subgroup generated by relations

$\{ i_U(w) = i_V(w) \quad w \in \pi_1(U \cap V) \}$

equality in $\pi_1(X, x_0)$

$S^2 \setminus k$ small disks



2 ways

$S^2 \setminus$ one ~~disk~~ disk $\cong D^2$

$S^2 \setminus k$ disks $\simeq D^2 \setminus (k-1)$ small disks

$\sim S_1 \vee S_1$

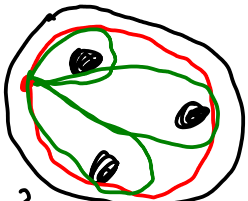
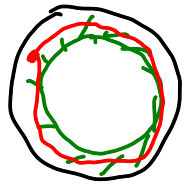
x_0

$\pi_1(D^2 \setminus (k-1)$ small disks$) = \pi_1(\underbrace{S^1 \vee S^1 \vee \dots \vee S^1}_{(k-1) \text{ times}})$

$\cong$ free group on $(k-1)$ generators

A different approach using van Kampen

$S^2 \setminus k$ disks $=$

$U =$  $\cup$  $= V$

$D^2 \setminus k$ small disk

D^2

$U \cap V \sim S^1$ $\qquad$ $U \sim \underbrace{S^1 \vee \dots \vee S^1}_{k\text{-times}}$ $\qquad$ $\pi_1(U)$ free group on k gen.

$V \sim *$ $\qquad$ $\pi_1(V) = \{1\}$

$\pi_1(U \cap V) \cong$ free group on 1 generator w.

Relation

$$\alpha_1 \alpha_2 \dots \alpha_k = \underline{1}$$

$$\pi_1(S^2 \setminus k \text{ disks}) \cong \frac{\text{gen. } \alpha_1 \alpha_2 \dots \alpha_k}{\text{relation } \alpha_1 \alpha_2 \dots \alpha_k = 1}$$

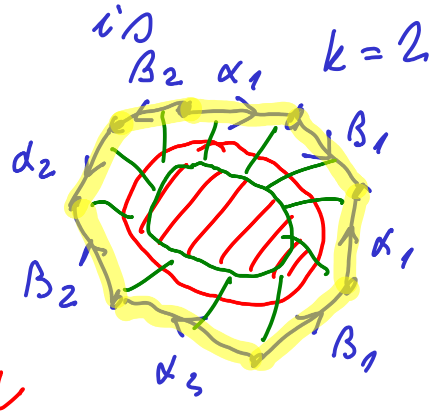
Previous description

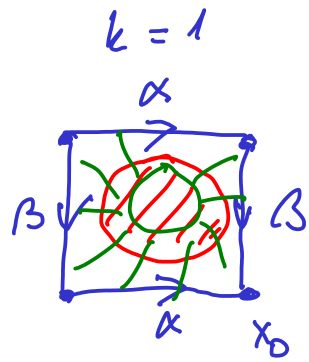
$\pi_1(S^2 - k \text{ disks}) \cong$ free group on $k-1$ gener.

Exercise 3. *Using Van Kampen Theorem compute the fundamental groups of the sphere with k handles.*

Sphere with k handles $k = 1$ torus

CW-model for this space is



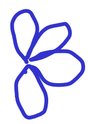


green

red

$X = U \cup V$ $U \cap V \sim S^1$ $\pi_1(U \cap V) \cong \mathbb{Z}$
w

$V \simeq D^2$ $\pi_1(V) \cong \{1\}$

$U \simeq$  $S^1 \vee S^1 \vee S^1 \vee S^1$ for $k = 2$

$\alpha_1 \quad \beta_1 \quad \alpha_2 \quad \beta_2$

$\pi_1(U) * \pi_1(V) \cong \pi_1(U)$ free group on $2k$ generators

relation : $w \xmapsto{i_U} \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1}$

$w \xmapsto{i_V} \underline{1}$

Relation is $\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1} = 1$

<u>Notice $k = 1$</u>

$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} = 1 \quad | \cdot \beta_1$

$\alpha_1 \beta_1 \alpha_1^{-1} = \beta_1 \quad | \cdot \alpha_1$

free $\underline{\alpha_1 \beta_1 = \beta_1 \alpha_1}$

$\pi_1(\text{torus})$ is an abelian group on 2 generators

Observation $T = S^1 \times S^1$

Generally it holds

$$\pi_n\left(X \times Y, (x_0, y_0)\right) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$$

$$(f, g) \quad (I^n, \partial I^n) \longrightarrow X \times Y, (x_0, y_0)$$

$$f \quad (I^n, \partial I^n) \longrightarrow (X, x_0)$$

$$g \quad (I^n, \partial I^n) \longrightarrow (Y, y_0)$$

Exercise 4. *Using Van Kampen Theorem compute the fundamental group of the sphere to which k Moebius bands are attached.*

$S^2 \setminus k$ small discs

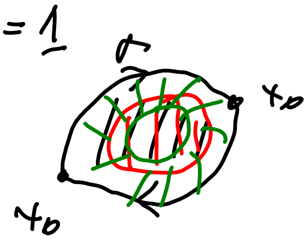
and we attach k Möbius bands

$k = 1$ — projective space of dim 2, $\mathbb{R}P^2$

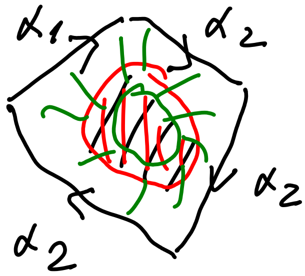
$k = 2$ — Klein bottle

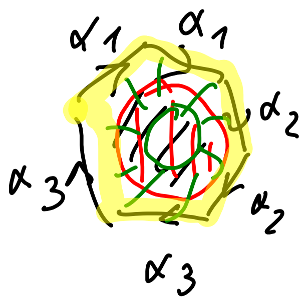
CW-model for it

$k = 1$



k general





green red

$X = U \cup V$

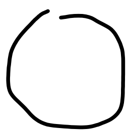
$V \simeq D^2 \qquad \pi_1(V) = \{1\}$

$U \simeq \underbrace{S^1 \vee S^1 \vee \ldots S^1}_{k \text{ times}} \qquad \alpha_1 \alpha_2 \ldots \alpha_k$

$U \cap V \simeq S^1 \qquad w$

$w \longmapsto \alpha_1^2 \alpha_2^2 \ldots \alpha_k^2$

$w \longmapsto 1$

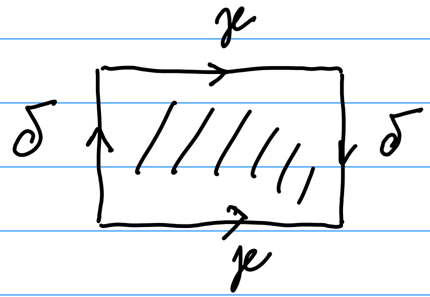


$S^2/\mathbb{Z}_2$

Relation is $\alpha_1^2 \alpha_2^2 \ldots \alpha_k^2 = 1$

$k = 1 \qquad \alpha_1^2 = 1 \qquad \underset{\mathbb{R}P^2}{\pi_1(X)} \cong \{-1, 1\} \cong \mathbb{Z}/2$

$k = 1$ Klein bottle



$\varkappa \, \delta$

$\varkappa \delta \varkappa^{-1} \delta = 1$

$\alpha_1 \alpha_2$

$\alpha_1^2 \alpha_2^2 = 1$

$\underbrace{\varkappa}_{\alpha_1} \varkappa \underbrace{\varkappa^{-1} \delta}_{\alpha_1} \underbrace{\varkappa^{-1} \delta}_{\alpha_2} = 1$

$\alpha_1 \longmapsto \varkappa \qquad \alpha_2 \longmapsto \varkappa^{-1} \delta$