b

Exercise 1. *If (X, A) is relative CW-complex such that there are no cells in dimension $\leq n$ in $X \setminus A$, then (X, A) is n-connected.*

$\pi_i(X, A) = 0 \qquad i \leq n$

$f: (D^i, S^{i-1}) \longrightarrow (X, A)$ we want to show that $[f] = 0$.

We use cellular appr. of f

$g: (D^i, S^{i-1}) \qquad g = f$ on S^{i-1}

g is cellular

$g((D^i)^{\varepsilon}) \subseteq A \cup X^{\varepsilon}$

$g \sim f$ rel S^{i-1}

$i \leq n$ no cells of dim $\leq i$ in $X \setminus A$

$g((D^i)^i) \subseteq A \cup X^i = A$

$\left[f \sim g \text{ rel } S^{i-1} \quad g(D^i) \subseteq A \Longleftrightarrow [f] = 0 \text{ in } \pi_i(X, A) \right]$

Exercise 2. *Let $[X,Y]$ denote a set of homotopy classes of maps from X to Y. If (X,x_0) is a CW-complex and Y is simply connected, then $[X,Y] \cong [(X,x_0),(Y,y_0)]$.*

$[X,Y]$ hom. classes of maps $X \to Y$ without fixed points

$[X,x_0;Y,y_0]$ hom. classes of map $(X,x_0) \to (Y,y_0)$ homotopy respect base points.

$\text{maps}(X,Y)/\sim$ .... $[X,Y]$ $=$ $\text{Map}(X,x_0;Y,y_0)/\sim$

$=$ $\text{Maps}(X,x_0;Y,y_0)/\sim_p$ .... $[X,x_0;Y,y_0]$

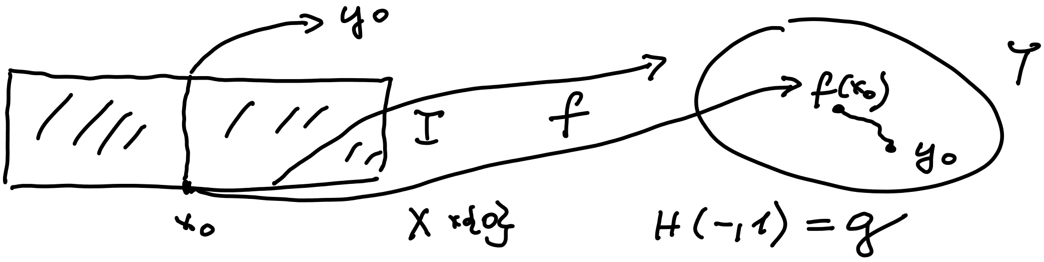
X CW-complex, Y is simply connected
$[X,Y] = [X,x_0;Y,y_0]$

$\text{map}(X,x_0;Y,y_0) \subseteq \text{maps}(X,Y)$

$\forall f: X \to Y \ \exists g:(X,x_0) \to (Y,y_0)$ $f \sim g$

HEP

$x_0 \hookrightarrow X$

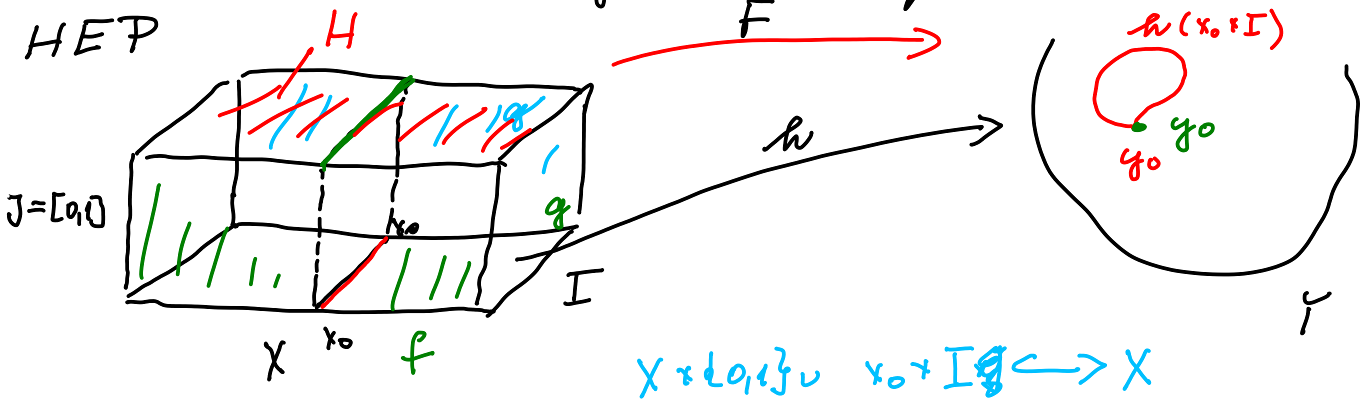


$H(-,1) = g$
$g(x_0) = y_0$
$f \sim g$

$f, g : (X,x_0) \to (Y,y_0)$ $f \sim g$ but the homotopy h does not keep fixed points

We find a new $H : f \sim g$ which is fixed on x_0

HEP



$X \times \{0,1\} \cup x_0 \times I \hookrightarrow X$

$F : X \times I \times J \to Y$, $H = F/X \times I \times \{1\}$

Exercise 3. *Show that the Hurewizc homomorphism is natural.*

$G : X \to Y$

$$\begin{tikzcd} \pi_n(X) \arrow[r, "h_X"] \arrow[d, "G_*"'] & H_n(X) \arrow[d, "G_*"] \\ \pi_n(Y) \arrow[r, "h_Y"] & H_n(Y) \end{tikzcd}$$

$f : S^n \to X$ $\quad h[f] = \underbrace{f_*}_{H_n(f)}(s) \in H_n(X)$ $\quad s \in H_n(S^n)$ generator

$f_* : H_n(S^n) \to H_n(X)$

$H_n(G)\, h_X[f] = h_Y(\pi_n(G) f)$

$L = H_n(G)\big(H_n(f)(s)\big) = H_n(G \circ f)(s)$

$R = h_Y[G \circ f] = H_n(G \circ f)(s)$ $=$

Exercise 4. *Show that the Hurewicz homomorphism commutes with connecting homomorphisms. It means: Let (X, A) be a pair. Show that the following diagram commutes:*

$$\begin{tikzcd}
{[f]} \in \pi_n(X, A, x_0) \arrow[r, "\partial"] \arrow[d, "h"'] & \pi_{n-1}(A, x_0) \arrow[d, "h"] \\
H_n(X, A) \arrow[r, "\partial_*"'] & H_{n-1}(A)
\end{tikzcd}$$

where ∂ is the boundary homomorphism, h is the Hurewicz homomorphism and ∂_* is the connecting homomorphism.

$f : (D^n, S^{n-1}) \longrightarrow (X, A)$

$s \in H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ generator

$h([f]) = f_*(s) \in H_n(X, A)$

$f_* \; H_n(D^n, S^{n-1}) \to H_n(X, A)$

$$\begin{tikzcd}
\underset{0}{H_n(D_n)} \arrow[r] & H_n(D^n, S^{n-1}) \arrow[r, "\partial", "\cong"'] \arrow[d, "f_*"'] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "(f/S^{n-1})_*"] & \underset{0}{H_{n-1}(D^n)} \\
& H_n(X, A) \arrow[r, "\partial"] & H_{n-1}(A) &
\end{tikzcd}$$

s generator $\quad \partial s = \bar{s}$ generator

$h_{X,A}[f] = f_*(s) \longmapsto \partial f_*(s) =$

$= (f/S^{n-1})_*(\bar{s})$

$\partial h_{X,A}[f] = h_A[\partial f]$

Exercise 5. *Show that the Hurewicz homomorphism $h : \pi_n(S^n) \to H_n(S^n)$ is an isomorphism.*

<u>Hurewicz Thm.</u> X is $(n-1)$-connected, then

$$\tilde{H}_i(X) = 0 \qquad i < n$$

$$H_n(X) = \pi_n(X)$$

$h : \pi_n(X) \to H_n(X)$ is an iso.

$X = S^n$ is $(n-1)$-connected

$h : \pi_n(S^n) \to H_n(S^n)$ is an iso.

$id : S^n \to S^n$ gives a generator of $\pi_n(S^n) \cong \mathbb{Z}$

$h[id] =$ generates in $H_n(S^n)$. ✓

$h[id] = (id)_*(s) = s$ ✓

Exercise 6. *Use Hopf fibration $S^1 \to S^3 \xrightarrow{f} S^2$ to compute $\pi_3(S^2)$.*

Long exact sequence of fibration

$\pi_1(S^1) \cong \mathbb{Z}$, $\pi_i(S^1) = 0$ $i \geq 2$

$\pi_1(S^3) = 0$, $\pi_2(S^3) = 0$, $\pi_3(S^3) \cong \mathbb{Z}$

$\pi_1(S^2) = 0$, $\pi_2(S^2) \cong \mathbb{Z}$

$$\underset{0}{\underset{\parallel}{\pi_3(S^1)}} \longrightarrow \pi_3(S^3) \underset{\cong}{\xrightarrow{f_*}} \pi_3(S^2) \longrightarrow \underset{0}{\underset{\parallel}{\pi_2(S^1)}} \to \dots$$

It gives that $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$.

Generator of $\pi_3(S^2)$

$id : S^3 \to S^3$ gives $[id]$ generator in $\pi_3(S^3)$

Image of this generator is

$$f_*[id] = [f].$$

This is a generator in $\pi_3(S^2)$.

f is called Hopf map

$$f : \underset{\mathbb{C}^2}{\underset{\cap}{S^3}} \longrightarrow S^2 = \mathbb{C}^2 \cup \{\infty\}$$

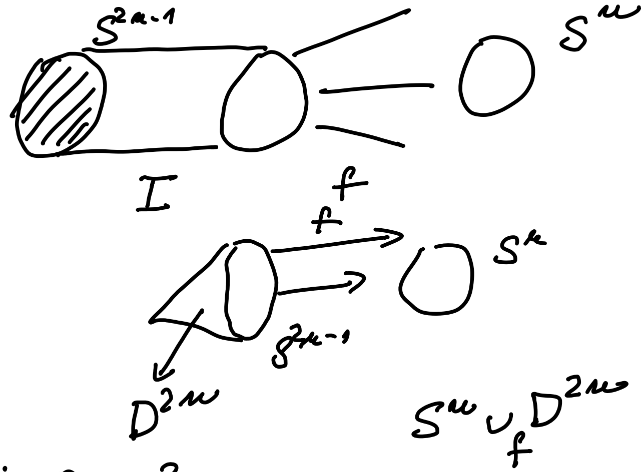
$$f(z_1, z_2) = \frac{z_2}{z_1}$$

Exercise 7. *(application) We know that deg(f) is an invariant of $[S^n, S^n] = \pi_n(S^n)$. Study $[S^{2n-1}, S^n] \cong \pi_{2n-1}(S^n)$ and describe its co called Hopf invariant $H(f)$.*

$f : S^{2n-1} \longrightarrow S^n$ $\qquad [f] \in \pi_{2n-1}(S^n)$

$S^n \cup_f D^{2n}$ $\qquad f : S^{2n-1} \to S^n$

$= C_f$



$S^n = e^0 \cup e^n$

$C_f = e^0 \cup e^n \cup e^{2n}$

$H^i(C_f) \cong \mathbb{Z}$ for $i = 0, n, 2n$

0 otherwise

$a \in H^n(C_f)$ generator

$b \in H^{2n}(C_f)$ generator

$a \cup a \in H^{2n}(C_f)$

$a \cup a = H(f) \cdot b$

number called Hopf invariant

Exercise 8. *What can we say in this case about Hopf inviariant for n odd and for n even?*

n odd

$$a \cup a = (-1)^{n \cdot n}\, a \cup a$$

$$a \cup a = - a \cup a$$

$$2\, a \cup a = 0 \quad \text{in } \mathbb{Z}$$

$$\Rightarrow \quad a \cup a = 0 \Rightarrow H(f) = 0.$$

Exercise 9. *Show that $H(f) = 1$ for the Hopf fibration $f : S^3 \to S^2$.*

$n = 2$ even $\quad S^{2n-1} \to S^n \quad S^3 \to S^2$

We show that in this case

$$C_f = D^4 \cup_f S^2 = \underline{\mathbb{C}P^2}$$

$$H^* \mathbb{C}P^2 = \mathbb{Z}[a] / a^3 = 0 \qquad a \in H^2(\mathbb{C}P^2)$$

generator of $H^2(\mathbb{C}P^2)$ is a

—— " —— $H^4(\mathbb{C}P^2)$ is $a \cup a \Rightarrow H(f) = 1$.

Recall CW-structure of $\mathbb{C}P^2$

$$\mathbb{C}P^2 \cong \mathbb{C}^3 / (z_1, z_2, z_3) \sim \lambda(z_1, z_2, z_3) \quad \lambda \neq 0$$

$$\cong S^5 / v \sim \lambda v \quad |\lambda| = 1$$

$$\cong \{(\underbrace{w_1, w_2}_{w}, \sqrt{1 - \|w\|^2}) \in \mathbb{C}^3 ; \|w\| \leq 1\} / (w, 0) \sim \lambda (w, 0)$$

$$\cong D^4 / w \sim \lambda w, \text{ for } \|w\| = 1$$

$$\cong D^4 \cup_f \mathbb{C}P^1$$

$$f : \partial D^4 \longrightarrow \mathbb{C}P^1 \cong S^2$$

$$(z_1, z_2) \longrightarrow \frac{z_2}{z_1}$$

Exercise 10. *Find a map f with Hopf invariant $H(f) = 2$.*

For any n even there is a map $f: S^{2n-1} \to S^n$ such that $H(f) = 2$.

James construction, X a CW-complex

$$J_2(X) = X \times X / \sim \qquad e \in X \text{ fixed point}$$

$$(x, e) \sim (e, x)$$

$J_2(S^n)$ has three cells in dim $0, n, 2n$.

$S^n = e^0 \cup e^n$ CW-structure

$$S^n \times S^n = \underbrace{e^0 \times e^0}_{0\text{-cell}} \cup \left.\begin{matrix} e^0 \times e^n \\ \cup\, e^n \times e^0 \end{matrix}\right\} n\text{-cells} \cup \underbrace{e^n \times e^n}_{2n\text{-cell}}$$

n-skeleton $J_2(S^n) = S^n$

$$J_2(S^n) = S^n \cup_f D^{2n} = C_f$$

We will compute $H(f)$.

$$p : S^n \times S^n \to J_2(S^n)$$

$$p^* \quad H^*(J_2(S^n)) \to H^*(S^n \times S^n)$$

$$\underbrace{1}_{H^0}, \ \underbrace{a_1, a_2}_{H^n}, \quad \underbrace{a_1 \cup a_2 = b_0}_{H^{2n}}$$

$1, \ a \in H^n(J_2(S^n))$, $b \in H^n(J_2(S^n))$

$$p^*(1) = 1$$

$$p^*(b) = a_1 \cup a_2 = b_0$$

$$p^*(a) = a_1 + a_2$$

$$a \cup a = H(f) \cdot b \qquad | \ p^*$$

$$p^*(a \cup a) = H(f)\, p^*(b)$$

$$p^*(a) \cup p^*(a) = H(f)\,(a_1 \cup a_2)$$

$$(a_1 + a_2) \cup (a_1 + a_2) = H(f)(a_1 \cup a_2)$$

$$\underbrace{a_1^2}_{=0} + a_1 \cup a_2 + a_2 \cup a_1 + \underbrace{a_2^2}_{=0} = H(f)(a_1 \cup a_2)$$

n even

$$2\, a_1 \cup a_2 = H(f)(a_1 \cup a_2)$$

$$\boxed{H(f) = 2}$$