Exercise 1. *If (X, A) is relative CW-complex such that there are no cells in dimension $\leq n$ in $X \setminus A$, then (X, A) is n-connected.*

Solution. Recall the definition of n-connectness of a pair. For $[f] \in \pi_i(X, A, x_0)$, $i \leq n$, use cell approximation of f: There is a cell map $q\colon (D^i, S^{i-1}, s_0) \to (X, A, x_0)$, such that $q \sim f$ relatively S^{i-1} and $q(D^i) \subseteq X^{(i)} = A$ since $X^{(-1)} = X^{(i)} = \cdots = X^{(n)} = A$. Note the following very useful criterion:

$[f] = 0$ in $\pi_i(X, A, x_0) \iff f \sim q$ relatively S^{i-1}, $g(D^i) = A$.

Thus $[f] = 0$ in our case, and we are done. □

Exercise 2. *Let $[X, Y]$ denote a set of homotopy classes of maps from X to Y. If (X, x_0) is a CW-complex and Y is simply connected, then $[X, Y] \cong [(X, x_0), (Y, y_0)]$.*

Exercise 3. *Show that the Hurewicz homomorphism is natural.*

Exercise 4. *Show that the Hurewicz homomorphism commutes with connecting homomorphisms. It means: Let (X, A) be a pair. Show that the following diagram commutes:*

$$\begin{array}{ccc} \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_n(X, A) & \xrightarrow[\partial_*]{} & H_{n-1}(A) \end{array}$$

where ∂ is the boundary homomorphism, h is the Hurewicz homomorphism and ∂_* is the connecting homomorphism.

Solution. Take $[f] \in \pi_n(X, A, x_0)$, that is $f\colon (D^n, D^{n-1}, s_0) \to (X, A, x_0)$. Then $\partial[f] = [f/S^{n-1}]$ and $h\partial[f] = h[f/S^{n-1}] = (f/S^{n-1})_*(b)$, where b is a generator in $H_{n-1}(S^{n-1})$. (We recall the definition of the Hurewicz homomorphism: if $g\colon S^{n-1} \to A$, $g_*\colon H_{n-1}(S^{n-1}) \to H_{n-1}(A)$ and $h[g] = g_*(b) \in H_{n-1}(A)$.)

Let $a \in H_n(D^n, S^{n-1})$ be a generator such that $\partial_* a = b$. We proceed using commutativity of the following diagram:

$$\begin{array}{ccc} H_n(D^n, S^{n-1}) & \underset{\cong}{\xrightarrow{\partial_*}} & H_{n-1}(S^{n-1}) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f/S^{n-1})_*} \\ H_n(X, A) & \xrightarrow[\partial_*]{} & H_{n-1}(A) \end{array}$$

Now $\partial_* h[f] = \partial_*(f_* a) = (f/S^{n-1})_*(\partial_* a) = h\partial[f]$ which concludes the proof. □

Exercise 5. *Show that the Hurewicz homomorphism $h : \pi_n(S^n) \to H_n(S^n)$ is an isomorphism.*

Exercise 6. *Use Hopf fibration $S^1 \to S^3 \to S^2$ to compute $\pi_3(S^2)$.*

Exercise 7. *(application) We know that deg(f) is an invariant of* $[S^n, S^n] = \pi_n(S^n)$. *Study* $[S^{2n-1}, S^n] \cong \pi_{2n-1}(S^n)$ *and describe its co called Hopf invariant* $H(f)$.

Solution. Have $f\colon \partial D^{2n} = S^{2n-1} \to S^n$ and $S^n \cup_f D^{2n}$. For $f \sim g$ we have $S^n \cup_f D^{2n} \simeq S^n \cup_g D^{2n}$, moreover $S^n \cup_f D^{2n} = C_f$ (the cylinder of f). For $n \geq 2$ we have $C_f = e^0 \cup e^n \cup e^{2n}$. Using cohomology: $H^*(C_f) = \mathbb{Z}$ for $* \in \{0, n, 2n\}$ and 0 elsewhere. Take $\alpha \in H^n(C_f)$ generator, we have cup product. Then $\alpha \cup \alpha \in H^{2n}(C_f)$ and for $\beta \in H^{2n}(C_f)$ we have $\alpha \cup \alpha = H(f)\beta$, where $H(f)$ is the Hopf invariant. □

Exercise 8. *What can we say in this case about Hopf inviariant for* n *odd and for* n *even? Thanks.*

Solution. Knowing $\alpha \cup \beta = (-1)^{|\alpha||\beta|}\beta \cup \alpha$ we see that $\alpha \cup \alpha = 0$. So for n odd Hopf invariant is zero.

For n even consider the Hopf fibration $S^1 \to S^3 \to S^2 = \mathbb{C}P^1$. For $\mathbb{C}P^2 = D^4 \cup_f \mathbb{C}P^1$ (recall how $\mathbb{C}P^n$ is built up from $\mathbb{C}P^{n-1}$) we have $C_f = \mathbb{C}P^2$ and $H^*(\mathbb{C}P^2) = \mathbb{Z}[\alpha]/\langle\alpha^3\rangle$, with $\alpha \in H^2$. The generator of H^4 is α^2. We get that $H(f) = 1$. □

Exercise 9. *Show that* $H(f) = 1$ *for the Hopf fibration* $f : S^3 \to S^2$.

Exercise 10. *Find a map* f *with Hopf invariant* $H(f) = 2$.

Solution. We study a space X with a basepoint e. Denote construction $J_2(X) = X \times X/\sim$, where $(x, e) \sim (e, x)$. Apply this idea to S^n. We get a projection $p\colon S^n \times S^n \to J_2(S^n)$. On the left we have one 0-cell, two n-cells and one 2n-cell, while on the right we have one of each. We get that $J_2(S^n)$ has to be a space of the form C_f, so $H^n(J_2) = \mathbb{Z}$ given by a and $H^{2n}(J_2) = \mathbb{Z}$ given by b and $H^n(S^n \times S^n) = \mathbb{Z} \oplus \mathbb{Z}$ (generators a_1, a_2) and $H^{2n}(S^n \times S^n) = \mathbb{Z}$ (with b_0). Now, $p^*\colon H^i(J_2) \to H^i(S^n \times S^n)$ and $p^*(a) = a_1 + a_2, p^*(b) = b_0$.

$$
\begin{aligned}
a^2 &= H(f)b \\
p^*(a^2) &= H(f)p^*(b) \\
(a_1 + a_2)^2 &= H(f)b_0 \\
(a_1^2 + a_1a_2 + a_2a_1 + a_2^2) &= H(f)a_1a_2 \\
2a_1a_2 &= H(f)a_1a_2 \\
H(f) &= 2
\end{aligned}
$$

because $b_0 = a_1a_2$ and by evenness of the dimension $a_1a_2 = a_2a_1$. □