Motivation

Literature:

Hovey – Model categories

Hirschhorn – Model categories and their localizations (reference book)

Dwyer, Spalinski – Homotopy theories and model categories

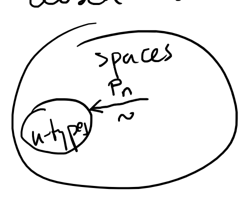
Quillen – Homotopical algebra (original)

Goerss, Jardine – Simplicial homotopy theory (sSet-oriented)

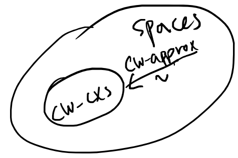
Now some motivation:

- they are a formalism to do homotopy theory
  → "abstract homotopy theory"
- examples:
  - topological spaces up to (weak) homotopy
  - simplicial sets up to (weak) homotopy
  - categories up to equivalence → ∞-categories
  - groupoids up to equivalence → ∞-groupoids
  - chain complexes (non-negatively graded, unbounded) up to homotopy equivalence / quasi-iso
  - n-categories ≃ homotopy n-types

spaces with homotopy groups concentrated in dimensions 0…n – not closed under categorical constructions



similarly:

- 

- projective resolution of a module / chain cx
- free G-spaces ⊆ G-spaces
  ⇒ homotopy theory of free G-spaces viewed as a structure on all

G-spaces (categorically better behaved)
that identifies any G-space with
the corresponding free G-space
(reflection / coreflection)

Definition

A model category is a category $\mathcal{M}$ equipped with three classes of morphisms

- $\xrightarrow{\sim}$ • $\mathcal{W}$ ... weak equivalences — things to be inverted
- $\rightarrowtail$ • $\mathcal{C}$ ... cofibrations — things that behave well w.r.t. to constructions like colimits
- $\twoheadrightarrow$ • $\mathcal{F}$ ... fibrations
- $\overset{\sim}{\rightarrowtail}$ • $\mathcal{W}\cap\mathcal{C}$ ... trivial cofibrations
- $\overset{\sim}{\twoheadrightarrow}$ • $\mathcal{W}\cap\mathcal{F}$ ... trivial fibrations — the same for limits

Axioms: An important notion: lifting property

$g \,\square\, h$ means

$$\begin{tikzcd} \cdot \arrow[r, "\forall a"] \arrow[d, "g"'] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "\forall b"'] \arrow[ur, dashed, "\exists \ell"] & \cdot \end{tikzcd}$$

$\forall a \forall b$: square commutes $\Rightarrow \exists \ell$: triangles commute

⚠ We have seen numerous examples:
surjective / injective / bijective mappings (in set)
injective modules & projective modules
serre / Hurewicz fibrations

$\mathcal{G}, \mathcal{H}$ two classes of morphisms: $\mathcal{G} \,\square\, \mathcal{H}$

$\Rightarrow \quad \mathcal{G}^{\square} \supseteq \mathcal{H} \iff \mathcal{G} \,\square\, \mathcal{H} \iff \mathcal{G} \subseteq {}^{\square}\mathcal{H}$

— factorization property:

$\mathcal{M} = \mathcal{H} \circ \mathcal{G}$ means: $\forall a\ \exists g \in \mathcal{G}, \exists h \in \mathcal{H} : a = h \circ g$

—

Now we are ready to state the axioms

MC1 (finite) limits and colimits exist in $\mathcal{M}$

MC2 2-out-of-3: $\begin{tikzcd} \cdot \arrow[rr, "f\circ g"] \arrow[dr, "g"'] & & \cdot \\ & \cdot \arrow[ur, "f"'] & \end{tikzcd}$ 2 of these in $\mathcal{W}$ $\Rightarrow$ so is 3rd
for $\mathcal{W}$

MC3 All $\mathcal{W}, \mathcal{C}, \mathcal{F}$ are closed under retracts

MC4 $\mathcal{C} \,\square\, (\mathcal{W}\cap\mathcal{F})$, $(\mathcal{W}\cap\mathcal{C}) \,\square\, \mathcal{F}$

MC5 $\mathcal{M} = (\mathcal{W}\cap\mathcal{F}) \circ \mathcal{C}$, $\mathcal{M} = \mathcal{F} \circ (\mathcal{W}\cap\mathcal{C})$

$\Big\}$ $(\mathcal{C}, \mathcal{W}\cap\mathcal{F})$ & $(\mathcal{W}\cap\mathcal{C}, \mathcal{F})$ form the so-called weak factorization systems

First properties

Retract argument:

$$f = g \circ h \quad \text{and} \quad f \,\square\, g \;\Rightarrow\; f \text{ is a retract of } h$$

Proof:

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "1"'] \arrow[ur, dashed, "\ell"] & \cdot \end{tikzcd} \rightsquigarrow \begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "f"'] & \cdot \arrow[r, "1"] \arrow[d, "h"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "\ell"'] & \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

$\square$

Lifting properties

- $\mathcal{W} \cap \mathcal{C} = {}^{\square}\mathcal{F}$, $(\mathcal{W} \cap \mathcal{C})^{\square} = \mathcal{F}$
-

Stability properties of the classes ${}^{\square}\mathcal{H}$, $\mathcal{G}^{\square}$ $\Rightarrow$ in particular for $\mathcal{C}, \mathcal{W}\cap\mathcal{C}$
$\mathcal{F}, \mathcal{W}\cap\mathcal{F}$

Meta-definition of the homotopy category, universal property

Cofibrant and fibrant replacement

Homotopy relation on maps, homotopy category

$\boxed{\text{Examples}}$ – DIFFICULT !