# Bousfield localizations

$\mathcal{M} \rightsquigarrow Ho(\mathcal{M}) \sim \mathcal{M}_{cf}/\text{htpy}$

What should a subcategory be ?

We want to limit fibrant / cofibrant objects.

(homotopy) injectivity $\doteq$ reflective subcat's — projectivity

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\sim"'] & X \\ B \arrow[ur, dashed] \end{tikzcd}$$

Example.

- $Ab \subseteq Gp$

$$\begin{tikzcd} \mathbb{Z} * \mathbb{Z} \arrow[r, "{(a,b)}"] \arrow[d] & A \\ \mathbb{Z} \oplus \mathbb{Z} \arrow[ur, dashed] \end{tikzcd}$$

$a * b \neq b * a$, $a+b = b+a$

- constant diagrams $D : \Delta \to \mathcal{M}$

$\mathcal{M} = Set$:
$$\begin{tikzcd} \Delta_m \arrow[r] \arrow[d] & D \\ \Delta_n \arrow[ur, dashed, "\exists !"'] \end{tikzcd} \quad \equiv \quad \begin{tikzcd} D_m \\ D_n \arrow[u, "\cong"'] \end{tikzcd}$$

$\mathcal{M}$ general: $\forall X$
$$\begin{tikzcd} \Delta_m \cdot X \arrow[r] \arrow[d] & D \\ \Delta_n \cdot X \arrow[ur, dashed, "\exists !"'] \end{tikzcd} \quad \equiv \quad \begin{tikzcd} \mathcal{M}(X, D_m) \\ \mathcal{M}(X, D_n) \arrow[u, "\cong"'] \end{tikzcd}$$

- diagrams $D : \mathcal{N} \to \mathcal{M}$ s.t. $D_n \xrightarrow{\cong} D_1 \times \cdots \times D_1$

$\mathcal{M} = Set$:
$$\begin{tikzcd} \mathcal{N}_1 + \cdots + \mathcal{N}_1 \arrow[r] \arrow[d] & D \\ \mathcal{N}_n \arrow[ur, dashed, "\exists !"'] \end{tikzcd} \quad \equiv \quad \begin{tikzcd} D_1 \times \cdots \times D_1 \\ D_n \arrow[u, "\cong"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{N}_1 \arrow[d] \\ \mathcal{N}_n \end{tikzcd} \quad \equiv \quad \begin{tikzcd} 1 \\ n \arrow[u] \end{tikzcd}$$

- diagrams $D : \mathcal{N} \to \mathcal{M}$ s.t. $D_n \xrightarrow{\cong} \{K, D_{n+1}\}$

$$\begin{tikzcd} \mathcal{N}_{n+1} \otimes K \arrow[r] \arrow[d] & D \\ \mathcal{N}_n \arrow[ur, dashed, "\exists !"'] \end{tikzcd} \quad \begin{tikzcd} \{K, D_{n+1}\} \\ D_n \arrow[u, "\cong"'] \end{tikzcd}$$

$\mathcal{V} = Top_* \,/\, sSet_*$

$K = S^1$

$\rightsquigarrow D_n \longrightarrow \Omega D_{n+1}$

$Hom(\mathcal{N}_{n+1} \otimes K, D) = \mathcal{V}(K, Hom(\mathcal{N}_{n+1}, D))$
$= \mathcal{V}(K, D_{n+1})$

- sheaves

$$\begin{tikzcd} \operatorname{colim} \mathcal{X}_{U_i} \arrow[r] \arrow[d] & D \\ \mathcal{X}_U \arrow[ur, dashed, "\exists !"'] \end{tikzcd} \quad \begin{tikzcd} \lim D_{U_i} \\ D_U \arrow[u, "\cong"'] \end{tikzcd}$$

$\mathcal{X} = Op(X)$

$U_i$ open covering of $U$ closed under intersections

Example. where htpy version is necessary

$\mathcal{M} = Top$:
$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X \\ D^{n+1} \arrow[ur, dashed, "\exists"'] \end{tikzcd}$$
$\exists$ unique up to higher htpies ?

$\pi_n X = 0$

$X$ is a homotopy $(n-1)$-type

$$\begin{tikzcd} map(S^n, X) \\ map(D^{n+1}, X) \arrow[u, "\sim"'] \end{tikzcd}$$

$\pi_n X = 0$

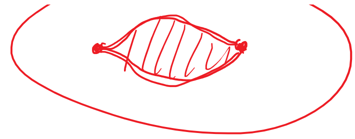
$\pi_{n+1} X = 0$

$\vdots$

more generally
$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ {} \arrow[ur] \end{tikzcd} \quad - \quad \begin{tikzcd} map(A, X) \\ {} \arrow[u] \end{tikzcd} \;\ldots$$

more generally

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ B \arrow[ur, dashed, "\exists!_h"'] & \end{tikzcd} \equiv \begin{tikzcd} \mathrm{map}(A,X) \\ \mathrm{map}(B,X) \arrow[u, "\sim"'] \end{tikzcd}$$



better:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \\ B \arrow[ur, dashed, "\exists!_h"'] & \end{tikzcd} \equiv \begin{tikzcd} \mathrm{map}(A,X) & L \arrow[l] \\ \mathrm{map}(B,X) \arrow[u, "\sim"'] & K \arrow[l] \arrow[u, tail] \arrow[ul, dashed] \end{tikzcd} \equiv \begin{tikzcd} K\times B +_{K\times A} L\times A \arrow[r] \arrow[d] & X \\ L\times B \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

cofibration (between cofibrant)

in Top $S^{n-1} \rightarrowtail D^n$

Will assume $\mathcal{M}$ left proper, cellular $\doteq$ cofibrantly generated with cofibrations effective mono's

**Definition.** Let $f: A \to B$ be a cofibration between cofibrant objects. We say that $W$ is f-local if it is fibrant and

$$f^*: \mathrm{map}(B,W) \xrightarrow{\sim} \mathrm{map}(A,W)$$

is a weak equivalence of simplicial sets. $\Big\}\ (\mathcal{J} \cup \{L^n B +_{L^n A} A^n \to B^n\})^{\square}$

$n=0$: $A \to B$

**Definition.** A map $g: X \to Y$ is an f-local equivalence if its cofibrant replacement $\tilde{g}: \tilde{X} \rightarrowtail \tilde{Y}$

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, equal] & \tilde{X} \arrow[r, "\sim"] \arrow[d, tail, "\tilde{g}"] & X \arrow[d, "g"] \\ 0 \arrow[r, tail] & \tilde{Y} \arrow[r, "\sim"] & Y \end{tikzcd}$$

$\Rightarrow$ f is an f-local equiv.

gives, for each f-local $W$, a w.e. $\tilde{g}^*: \mathrm{map}(\tilde{Y},W) \xrightarrow{\sim} \mathrm{map}(\tilde{X},W)$

(any two related by a zig-zag of w.e.'s of such $\Rightarrow$ independent of choice)

**Definition.** An f-localization of $X$ is an f-local equivalence

$$j: X \longrightarrow \hat{X} \quad \text{with } \hat{X} \text{ f-local.}$$

**Aim.** Construct an "f-local" model structure in which:

- f-local = fibrant; fibrations are complicated BUT cofibrations of $\mathcal{M}$
- f-local equivalence = weak equivalence
- f-localization = fibrant replacement = "reflection"

$\to$ better: $\mathrm{Id}: \mathcal{M} \rightleftarrows \mathcal{M}^{f\text{-local}} : \mathrm{Id}$

preserves cof. & w.e.

is interpreted as $\mathcal{M}_{cf} \underset{f\text{-localization}}{\rightleftarrows} \mathcal{M}^{f\text{-local}}_{cf}$

$$\mathcal{M}(B^*, W) = \mathrm{Map}(B,W)$$

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Map}(B,W) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Map}(A,W) \end{tikzcd}$$

$$\begin{tikzcd} L^n B +_{L^n A} A^n \arrow[r] \arrow[d] & W \\ B^n \arrow[ur, dashed] & \end{tikzcd}$$

$n=0$: $\begin{tikzcd} A \arrow[r] \arrow[d] & W \\ B \arrow[ur, dashed] & \end{tikzcd}$

Cellular f-localization functor

$n=0$ : $A \to W$, $A \downarrow B$, $B \dashrightarrow W$

# Constructing f-localization functor

- we want to use SOA w.r.t.

$\overline{\Lambda f} = J \cup \Lambda f$

generating triv. cof.

Reedy cof. $f^*: A^* \rightarrowtail B^*$ frame on $f: A \rightarrowtail B$

$\Lambda f = \{ L^n B +_{L^n A} A^n \to B^n \mid n \}$

horns on $f$

$$\begin{array}{ccc} A^* & \xrightarrow{f^*} & B^* \\ \sim\downarrow & & \downarrow\sim \\ cst A & \xrightarrow[f]{} & cst B \end{array}$$

because $\overline{\Lambda f} \,\square\, \{W \to *\} \iff W$ is f-local

Important ingredience:

$\overline{\Lambda f}$ - cell $\Rightarrow$ f-local equivalence

the result of the SOA

$X \longrightarrow L_f X \longrightarrow *$

$\overline{\Lambda f}$-cell $\in (\overline{\Lambda f})^{\square} \iff L_f X$ is f-local

$\cap$

f-local equiv

$\Rightarrow$ then $L_f X$ is an f-localization

$n=0$ : $f: A \to B$

$\overline{\Lambda f} = J \cup \{ L^n B +_{L^n A} A^n \to B^n \}$ $n$ arb: $\mathrm{Map}(L^n B +_{L^n A} A^n, W) \xleftarrow{?} \mathrm{Map}(B^n, W)$

consists of cofibrations + f-local equivalences and these are closed under coproducts, pushouts and transfinite compositions

$j: A \to B$ in $J$ is f-local equiv.

$j^*: \mathrm{Map}(\tilde{B}, W) \xrightarrow{\sim} \mathrm{Map}(\tilde{A}, W)$

($\tilde{A} \overset{\sim}{\rightarrowtail} B$ induces triv. fib.)

coproducts: $j_i: A_i \to B_i$ cof. + f-local equiv's coproducts: $j_i : A_i \to B_i$ cof. + f-local equivs

$$\mathrm{Map}(\tilde{B}_i, W) \xrightarrow{\sim} \mathrm{Map}(\tilde{A}_i, W)$$

$$\Downarrow$$

$$\prod \mathrm{Map}(\tilde{B}_i, W) \xrightarrow{\sim} \prod \mathrm{Map}(\tilde{A}_i, W)$$

$$\mathrm{Map}(\textstyle\sum \tilde{B}_i, W) \xrightarrow{\sim} \mathrm{Map}(\textstyle\sum \tilde{A}_i, W)$$

pushouts:
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\sim_f"'] & C \arrow[d] \\ B \arrow[r] & D \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \rightsquigarrow \underbrace{\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d, tail] & \tilde{C} \arrow[d, tail] \\ \tilde{B} \arrow[r] & \tilde{D} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}}_{\text{cof obj.}} \quad / \ \mathrm{Map}(-, W)$$

$$\begin{tikzcd} \mathrm{Map}(\tilde{A}, W) & \mathrm{Map}(\tilde{C}, W) \arrow[l] \\ \mathrm{Map}(\tilde{B}, W) \arrow[u, "\sim"] & \mathrm{Map}(\tilde{D}, W) \arrow[l] \arrow[u, "\sim"'] \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \Rightarrow C \to D \text{ is an f-loc. eq.}$$

since $A \to B$ is f-loc. eq.

transf. comp.
$$\begin{tikzcd} A_0 \arrow[r, "\sim_f"] \arrow[rrr, bend right] & A_1 \arrow[r, "\sim_f"] & \cdots & A_\infty \end{tikzcd}$$

$$\tilde{A}_0 \rightarrowtail \tilde{A}_1 \rightarrowtail \cdots \qquad / \ \mathrm{Map}(-, W)$$

$$\begin{tikzcd} \mathrm{Map}(\tilde{A}_0, W) & \mathrm{Map}(\tilde{A}_1, W) \arrow[l, two heads, "\sim"'] & \cdots \arrow[l, two heads, "\sim"'] & \mathrm{Map}(\tilde{A}_\infty, W) \arrow[lll, bend left, "\sim"] \end{tikzcd}$$

$\Rightarrow$ SOA w.r.t $\overline{\Lambda f}$ gives functorial f-localization but NOT arbitrary factorizations

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\sim_f"'] & * \\ L_f X \arrow[ur, "\in (\overline{\Lambda f})^\square"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ \bullet \arrow[ur, "\in (\overline{\Lambda f})^\square"'] & \end{tikzcd}$$

$\neq$ fibrations in the localization

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<u>local model structure</u> $\quad L_f M = M +$ local cof, fib, w.e.

local cofibrations = cofibrations

local weak equiv = f-local equivalences

local weak equiv = $f$-local equivalences
local fibrations = (cofib + $f$-local eq.)$^\square$

**Theorem.** $\mathcal{M}$ left proper cellular model cat.
$\Rightarrow$ • $L_f\mathcal{M}$ is a model category, cellular
• fibrant objects = $f$-local objects
• $L_f\mathcal{M}$ is left proper
• $\mathcal{M}$ simplicial $\Rightarrow$ $L_f\mathcal{M}$ simplicial

**Remark.** $\mathcal{M}, L_f\mathcal{M}$ right proper

$\Rightarrow$ $\begin{array}{c} X \\ p\downarrow \\ Y \end{array}$ is a local fibration $\Leftrightarrow$ $p$ is a fibration in $\mathcal{M}$

and the map into the pullback in
$$\begin{tikzcd} X \arrow[r] \arrow[dr] \arrow[dd, "p"'] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_fX \arrow[d] \\ & Y \arrow[r] & L_fY \end{tikzcd}$$
is a w.e. in $\mathcal{M}$

$$\left( \begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & Y & \end{tikzcd} \qquad \begin{array}{l} p \text{ is a local fib.} \\ \Leftrightarrow p' \text{ is a local fib.} \end{array} \right)$$

Main ingredient:
construct the set of gen. triv. cof.
$\rightsquigarrow$ find a cardinal $\lambda$
and take all relative $I$ cell complexes of "size $< \lambda$" that are at the same $f$-local equiv's — Bousfield cardinality argument

# Homotopy uniqueness of diagonals

$M$ simplicial model cat
e.g. $\mathrm{map}(B,X)_n = \mathrm{Top}(B\times|\Delta^n|, X)$
plus w.e. if i/p is one

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{map}(B,X) \arrow[d, "\mathrm{map}(i,p)_r", "\text{simplicially}"'] \\ \mathrm{map}(A,X)\times_{\mathrm{map}(A,Y)}\mathrm{map}(B,Y) \end{tikzcd}$$

fibres are $\underbrace{\mathrm{map}_{A/M/Y}(B,X)}$
more precisely later

commutative squares ( i,p fixed)

$A\otimes L +_{A\otimes K} A\otimes K$
$\downarrow$
$B\otimes L$

generally replace by $\mathrm{Map}(B,X) = M(B^*,X)$ etc.

cof. repl. cst $B \in [\Delta, M]_R$
$B^0 = B$
$B^1 = \mathrm{Cyl}\,B$
etc.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, "\mathrm{in}"'] & \mathrm{Map}(B,X) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \equiv \begin{tikzcd} L^n i^* \cdot \arrow[r] \arrow[d, "\partial_n i^*"'] & X \arrow[d, "\sim", "p"'] \\ \Delta^n \cdot B^* \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$B^n =$

assume
$A,B$ cofibrant
$X,Y$ fibrant

$A^* \rightarrowtail B^*$
Reedy cofibration

$L^! i^*$ $L^1 B \subseteq$ $B^1$ $A^1$

Differently: $B \in (A/M/Y)_c$ , $X \in (A/M/Y)_f$

$0$ in $A/M/Y$ is $1$
$A = A \to Y$
$A\to Y = Y$

$A \to M \to Y$
fixed

$$\begin{tikzcd} & \mathrm{Map}(B,X) \arrow[dl] \arrow[dr] & \\ * \simeq \mathrm{Map}(A,X) & & \mathrm{Map}(B,Y) \simeq * \end{tikzcd}$$

(since $A$ is initial) (since $Y$ is terminal)

and one of them is a w.e.

Application: Assume that $M$ is left proper and cofibrantly generated by sets $I, J$.

$$\begin{tikzcd} \tilde A \arrow[r, "\sim"] \arrow[d, "\tilde k"'] & A \arrow[d, "k"] \\ \tilde B \arrow[r, "\sim"'] & B \end{tikzcd}$$

cofibration

$\rightsquigarrow$ set $\tilde I$ of cofibrations between cofibrant objects

Proposition. $f$ fibration. Then $\tilde k \,\square\, f \Rightarrow k \,\square\, f$.

Proof.

$$\begin{tikzcd} \tilde A \arrow[r, "\sim"] \arrow[d, "\tilde k"'] & A \arrow[r] \arrow[d, "k"] \arrow[dl] & X \arrow[d, "f"] \\ \tilde B \arrow[r, "\sim"'] & B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr] \arrow[d] \arrow[ddr, "k"] & & X \arrow[dd, "f"] \\ P \arrow[dr, "\sim"'] \arrow[urr, dashed] & & \\ & B \arrow[r] & Y \end{tikzcd}$$

pushout

$P \longrightarrow X$

since $M$ is left proper

since $\mathcal{M}$ is left proper pushout

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\sim"'] & X \\ B & \end{tikzcd}$$

cof   fib.   in $A/\mathcal{M}/Y$

$$\mathrm{Map}(P,X) \neq \emptyset$$

$$\uparrow \sim$$

$$\mathrm{Map}(B,X) \neq \emptyset \qquad \square$$