Bousfield localization

$$\begin{tikzcd} X \arrow[r, "u_f"] \arrow[d, "u"'] & L_f X \arrow[d, "L_f u"] \\ Y \arrow[r, "u_f"] & L_f Y \end{tikzcd}$$

$u: X \to Y$ an f-local equiv
iff $L_f u: L_f X \to L_f Y$ weak equivalence

Theorem (Whitehead)

A map $u: X \longrightarrow Y$ between f-local objects is an f-local equiv iff it is a weak equivalence.

"$\Leftarrow$": $u^*: \mathrm{Map}(Y, W) \xrightarrow[\sim]{} \mathrm{Map}(X, W)$

Proof.

$$\begin{tikzcd} \tilde{X} \arrow[r, "r", "\sim"'] \arrow[d, "\tilde{u}"'] & X \arrow[d, "u"] \\ \tilde{Y} \arrow[r, "\sim", "s"'] \arrow[ur, "\sim", "v"'] & Y \end{tikzcd}$$

$\tilde{u}^*: \mathrm{Map}(\tilde{Y}, X) \xrightarrow{\sim} \mathrm{Map}(\tilde{X}, X) \ni r$

Ψ

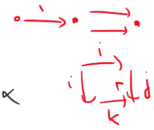
$\tilde{u}^*: \mathrm{Map}(\tilde{Y}, Y) \xrightarrow{\sim} \mathrm{Map}(\tilde{X}, Y)$

$s \,,\, uv$ $\qquad$ $s\tilde{u} \,,\, uv\tilde{u}$

$\qquad\qquad\qquad = ur$

$\square$

cellular model categories: assume that $\mathcal{M}$ is cofibrantly generated with a set I of generating cofibrations that are effective mono's (= equalizers). Then any relative I-cell complex



$$X = X_0 \to X_1 \to \cdots \to X_\lambda = Y$$

with cells

$$\begin{tikzcd} \sum\limits_{t \in T_\alpha} A_t \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ \sum\limits_{t \in T_\alpha} B_t \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X_{\alpha+1} \end{tikzcd}$$

admits a reasonable notion of a subcomplex – namely a collection of subsets $\overline{T}_\alpha \subseteq T_\alpha$ that give via subpushouts

$$\begin{tikzcd} \sum\limits_{\overline{T}_\alpha} A_t \arrow[r] \arrow[dr] \arrow[dd] & \sum\limits_{\overline{T}_\alpha} B_t \arrow[dr] \arrow[dd] & & & \\ X = \overline{X}_0 \arrow[r] \arrow[dd, "="] & \overline{X}_1 \arrow[r] \arrow[dd] & \cdots \arrow[r] & \overline{X}_\lambda = \overline{Y} \arrow[dd] & \\ \sum\limits_{T_\alpha} A_t \arrow[r] \arrow[dr] & \sum\limits_{T_\alpha} B_t \arrow[dr] & & & \\ X = X_0 \arrow[r] & X_1 \arrow[r] & \cdots \arrow[r] & X_\lambda = Y & \end{tikzcd}$$

IF they exist
= condition on $\overline{T}_\alpha$'s

those then exist uniquelly and are effective monos

We call $\left| \sum_\alpha T_\alpha \right|$ the size of the complex Y (relative to X)
= number of cells

– compactness of A: $\operatorname{colim} \mathcal{M}(A, X_i) \xrightarrow{\cong} \mathcal{M}(A, \operatorname{colim} X_i)$
(κ-compactness)

for X_i the collection of all subcomplexes of a complex $X = \operatorname{colim} X_i$ of size $< \kappa$

Hirschhorn $\leq \kappa$

Since we assume I to consist of effective monos, this is always injective, i.e. compactness means

$$\begin{tikzcd} & X_i \arrow[r] \arrow[d] & X_j \arrow[dl] \\ A \arrow[ur, dashed, "\exists"] \arrow[r] & X & \end{tikzcd}$$

$\Longrightarrow$ A is small since the cells of X_i lie in some step of the transfinite composition lie in some step of the transfinite composition

Definition. $\mathcal{M}$ is cellular if it is

- cofibrantly generated by sets I, J
- domains and codomains of I are compact $\Rightarrow$ all cofibrant objs compact
- domains and codomains of J are ~~small w.r.t. cofibrations~~ compact
- cofibrations are effective monos

Example. If $\mathcal{M}$ is locally presentable, 2 and 3 are automatic.
($\mathcal{M}$ combinatorial)

Important ingredient:

- Let X be a set of relative cell complexes with compact domains and $f: X \longrightarrow Y$ a map. By applying the SOA
cell complex
factor it as $f: X \xrightarrow{i} \hat{X} \xrightarrow{p} Y$. Then size $\hat{X}$ = size X provided that it is big enough.
  - bounded set of squares
$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ B_i \arrow[r] & Y \end{tikzcd}$$
$$\underbrace{A_i \dashrightarrow (X_\alpha)_\beta}_{\text{set of such up to iso}} \longrightarrow X_\alpha$$
  - the size of X_α does not increase

- Application:
  - X cell complex $\Rightarrow$ $\mathrm{Cyl}\,X$ cell complex of equal size
(SOA w.r.t. I applied to $X+X \longrightarrow X$)
  - $f: A \longrightarrow B$ inclusion of cell complexes
SOA w.r.t. $\overline{\Lambda f} = J \cup \{L^n B +_{L^n A} A^n \longrightarrow B^n\}$ applied to $X \longrightarrow *$
gives the f-localization $X \longrightarrow L_f X$ of equal size

Properties of the f-localization:

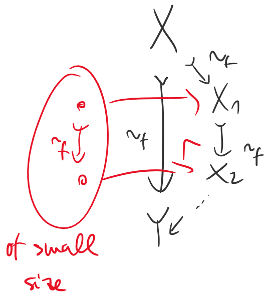
- it respects intersection of subcomplexes: $L_f(\bigcap X_i) = \bigcap L_f X_i$
(follows from the way that SOA works)
- for each subcomplex $Z \subseteq L_f X$ there is a minimal subcomplex $W \subseteq X$ s.t. $Z \subseteq L_f W$ and size W = size Z (above some μ ...)
- it respects directed unions of subcomplexes $\operatorname{colim} L_f X_i = L_f \operatorname{colim} X_i$

- it respects directed unions of

Theorem. (Bousfield Cardinality Argument)

Every inclusion of cell complexes that is an f-equivalence can be expressed as a relative cell complex w.r.t. small inclusions of cell complexes that are f-equivalences.

Proof.



— by maximality principle, enough to extend

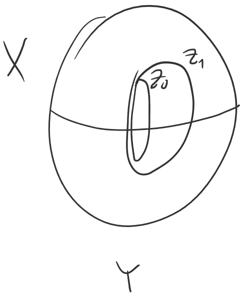
Whitehead theorem

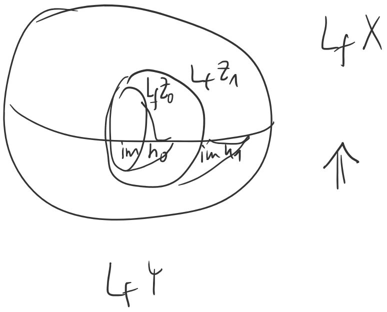
$$\begin{tikzcd} X \arrow[r] \arrow[d, "\sim_f"'] & L_f X \arrow[d, "\sim"] \\ Y \arrow[r] & L_f Y \end{tikzcd}$$

$\Longrightarrow$ there exists a deformation

$h: \mathrm{Cyl}\, L_f Y \longrightarrow L_f Y$

constant on $\mathrm{Cyl}\, L_f X$





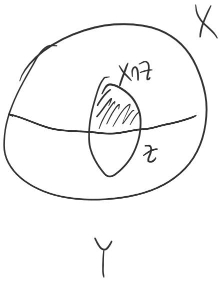
$h_0 = h|_{\mathrm{Cyl}\, L_f Z_0}$

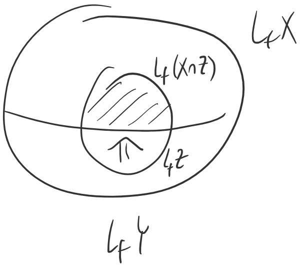
$h_1 = h|_{\mathrm{Cyl}\, L_f Z_1}$

$\vdots$

$$\begin{tikzcd} Z_0 & \mathrm{Cyl}\, L_f Z_0 \arrow[dr] & L_f Z_0 \\ \cap & \cap & \cap \\ Z_1 & \mathrm{Cyl}\, L_f Z_1 \arrow[dr] & L_f Z_1 \\ \cap & \cap & \cap \\ \vdots & \vdots & \vdots \\ \cap & & \\ Z & & \end{tikzcd}$$

$\mathrm{Cyl}\, L_f Z \longrightarrow L_f Z$ deformation onto $L_f(X \cap Z)$





small
$$\begin{tikzcd}
X\cap Z \arrow[r] \arrow[d, "\sim_f"'] & X \arrow[d, "\sim_f"] \arrow[dd, bend left=60, "\sim_f"] \\
Z \arrow[r] & X\cup Z \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] \\
 & Y
\end{tikzcd}$$
extension

$$\begin{tikzcd}
L_f(X\cap Z) \arrow[d, "\sim"'] \\
L_f Z
\end{tikzcd}$$

$\boxed{\text{I}}$

Theorem. (Left Bousfield localization)

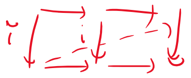
M left proper cellular f inclusion of cell complexes
$\Rightarrow$ f-local model structure exists and is left proper cellular

Proof. • recognizing f-local fibrations:

$\mathcal{J}_f = \mathcal{J} \cup \mathcal{J}_f^0$

$$\mathcal{F}_f = (\mathcal{W}_f \cap \mathcal{I})^{\square} \overset{\text{left proper}}{=} \mathcal{J}^{\square} \cap (\mathcal{W}_f \cap \mathcal{I})_{\text{cell}}^{\square} \overset{\text{BCA}}{=} \mathcal{J}^{\square} \cap (\mathcal{J}_f^0)^{\square} = \mathcal{J}_f^{\square}$$

$\uparrow$ inclusions of small subcomplexes that are f-loc. eq.



$$\Rightarrow \mathcal{W}_f \cap \mathcal{I} \overset{?}{=} {}^{\square}((\mathcal{W}_f \cap \mathcal{I})^{\square}) = {}^{\square}(\mathcal{J}_f^{\square})$$

• recognizing f-local trivial fibrations $\overset{?}{=}$ trivial fibrations $\mathcal{I}^{\square}$

"$\Leftarrow$": $\mathcal{I}^{\square} = \mathcal{W} \cap \mathcal{I}^{\square} \subseteq \mathcal{W}_f \cap \mathcal{J}_f^{\square}$

"$\Rightarrow$": uses retract argument (part of a more technical recognition theorem for cofibrantly generated model cat's)

$$\begin{tikzcd}
 & \bullet \arrow[dr, two heads, "p", "\sim"'] & \\
\bullet \arrow[ur, "i"] \arrow[rr, "u"'] & & \bullet
\end{tikzcd}$$

$$\underset{\mathcal{W}_f \cap \mathcal{J}_f^{\square}}{\overset{u}{\cap}}$$

$u \in \mathcal{W}_f$, $p \in \mathcal{W} \subseteq \mathcal{W}_f$ $\Rightarrow$ $i \in \mathcal{W}_f \cap \mathcal{I} = {}^{\square}(\mathcal{J}_f^{\square})$

$\Rightarrow$ u retract of $p \in \mathcal{W} \cap \mathcal{I}^{\square}$

- $\mathcal{J}_f$-cell $\subseteq \mathcal{W}_f$
- left proper easy
- cellular clear

$\overset{\text{mor}\ \ \text{mor}}{\square}$ $\underset{\text{mor}\ \ \text{obj}}{\triangledown}$

$\boxed{\text{I}}$

Fibrant objects lie in $\mathcal{J}_f^{\triangledown} \subseteq \overline{\Lambda f}^{\triangledown}$ $\Rightarrow$ they are f-local.

In the opposite direction:

$$\begin{tikzcd}
K \arrow[r] \arrow[d, tail] & X \\
L \arrow[ur, dashed]
\end{tikzcd}$$

$\mathcal{J}_f^0 \ni$ ($K \rightarrowtail L$)

$\leftarrow$ f-local

$\exists$ up to htpy that can be removed

$$\begin{tikzcd}
\mathrm{Map}(K,X) \\
\mathrm{Map}(L,X) \arrow[u, "\sim"']
\end{tikzcd}$$

Theorem.

Fibrant objects are exactly the f-local objects.

We will also describe fibrations between fibrant objects.

Proposition.

$$\begin{tikzcd} X \arrow[rr, "u", "\sim"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

p is an f-local fibration

$\Leftrightarrow$ q is an f-local fibration

Proof. We start with the easier "$\Leftarrow$".

$$\begin{tikzcd} & K \arrow[r] \arrow[d] & X \arrow[r, "u", "\sim"'] \arrow[d] & Y \arrow[dl] \\ \mathcal{J}_f \ni & L \arrow[r] \arrow[ur, dashed] & Z & \end{tikzcd}$$

in $\mathcal{M}$ has a simple interpretation

in $K/\mathcal{M}/Z$:

$$\begin{tikzcd} & X \arrow[d, "u", "\sim"'] \\ L \arrow[ur, dashed] & Y \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Map}(L,X) \arrow[d, "\sim"] & \text{non-empty} \\ \mathrm{Map}(L,Y) & \text{non-empty} \arrow[u, Rightarrow] \end{tikzcd}$$

The direction "$\Rightarrow$" reduces to u being a trivial fibration (Brown)

K cell complex $\longrightarrow$

$$\begin{tikzcd} & X \arrow[d, "u", "\sim"'] \\ K \arrow[r] \arrow[d, tail] \arrow[ur, dashed, "\exists"] & Y \arrow[d] \\ L \arrow[r] & Z \end{tikzcd}$$

$\rightsquigarrow$ again can be turned to $K/\mathcal{M}/Z$.

$\square$

Theorem. A map $p: X \to Y$ between f-local objects is an f-local fibration $\Leftrightarrow$ it is a fibration.

↖ f-local fibrations between general objects ???

Proof.

$$\begin{tikzcd} X \arrow[rr, tail, "\sim_f", "u"'] \arrow[dr, "p"'] & & Z \arrow[dl, "q", "\mathcal{J}_f"'] \\ & Y & \end{tikzcd}$$

$\Rightarrow$ Z also f-local

$\overset{\text{Whitehead}}{\Rightarrow}$ u a weak equivalence

$\Rightarrow$ p also f-local fibration.

H: $\underbrace{\mathcal{M}/L_f\mathcal{M}}_{?}$ right proper

$\square$

Some other formalisms ... fibrations are only important between fibrant objects

$=$

prezentace grup $\longrightarrow$ grupy

spektra

Postnikovovy věže

homologické lokalizace

rac. htpická teorie $\xrightarrow{\sim_Q}$ CDGA

LDGA

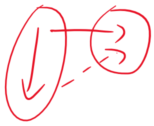
kategorie $\longrightarrow$ grupoidy

$N: \mathrm{Cat}_T \xrightarrow{\sim_Q} s\mathrm{Set}_Q$

Reminder

20. ledna 2021 15:41

M cof. gen., I set of gen. cof.



$f : A \to B$ inclusion of I-cell complexes

$\Lambda f = \{ L^n B \sqcup_{L^n A} A^n \to B^n \}$

$\overline{\Lambda f} = \mathcal{J} \cup \Lambda f$

$f^* : Map(B, W) \xrightarrow{\sim} Map(A, W)$

$\overline{\Lambda f}^{\boxslash} =$ f-local objects

f-local equivalences : $u : X \to Y$ s.t. $\tilde{u} : \tilde{X} \rightarrowtail \tilde{Y}$ induces w.e. $\tilde{u}^* : Map(\tilde{Y}, W) \xrightarrow{\sim} Map(\tilde{X}, W)$ $\forall W$ f-local

$u \in W_f \cap I \ldots u \boxslash \{\text{f-local}\}$

Important: $\overline{\Lambda f} \subseteq I \cap W_f$ and the latter is closed under coproducts, pushouts, transfinite compositions, retracts. ! M left proper

$\Rightarrow$ SOA gives f-localization $\underbrace{X \longrightarrow L_f X}_{\substack{\text{relative } \overline{\Lambda f}\text{-cell complex} \\ \Rightarrow \ \in I \cap W_f}} \underbrace{\longrightarrow *}_{\text{f-local}}$ f-localization