Left Bousfield localizations

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The model category structure determined by cofibrations and fibrant objects $\Rightarrow$ possible to describe localizations dually via local objects (but existence more subtle)

Idea:

$$\begin{tikzcd} A \arrow[d, "\sim\ f"] \\ B \end{tikzcd} \iff \begin{tikzcd} 0 \arrow[r, hook] \arrow[rr, bend left, hook] & A^c \arrow[r, two heads, "\sim"] \arrow[d, "f^c\ \sim"] & A \arrow[d, "\sim\ f"] \\ & B^c \arrow[r, two heads, "\sim"] & B \end{tikzcd} \iff \forall X \in \mathcal{M}_f : \ Ho(\mathcal{M})(B^c, X) \xrightarrow[\cong]{(f^c)^*} Ho(\mathcal{M})(A^c, X)$$

$B^c \xrightarrow{\sim} B$ fixed

$Ho(\mathcal{M})(B^c,X) = \mathcal{M}(B^c,X)/\text{left htpy}$ — determined by cofibrations

Examples. • Joyal model structure on simplicial sets

cofibrations = monos

fibrant objects = quasi-categories

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd} \qquad 0<i<n \qquad \text{inner horns}$$

• Kan model structure on simplicial sets

cofibrations = monos

fibrant objects = Kan complexes

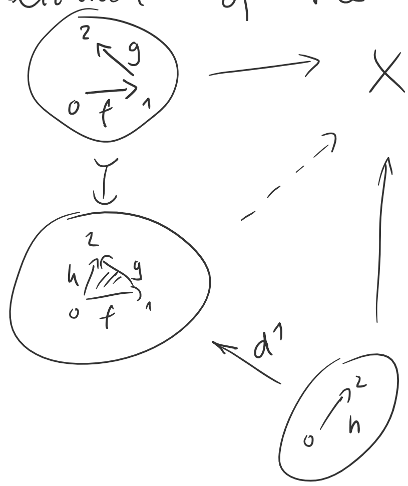
$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd} \qquad 0\le i\le n \qquad \text{all horns}$$

$$\begin{tikzcd} & Ex^\infty X \arrow[d, equal] \\ X \arrow[r] \arrow[d, "\sim\ f"] & LX \arrow[d, "\sim"] \\ Y \arrow[r] & LY \arrow[d, equal] \\ & Ex^\infty Y \end{tikzcd}$$

$f^* : [Y,Z] \xrightarrow{\cong} [X,Z]$ — $Z$ Kan complex

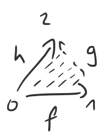
$\Rightarrow$ left Bousfield localization of the Joyal model structure

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \\ & \Delta^1 \arrow[ul, "d^1"] \arrow[uu, dashed] \end{tikzcd}$$



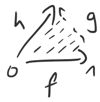

composition $h = g \circ f$ $\rightsquigarrow$ quasi category

$\Lambda^2_0$

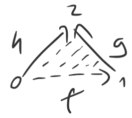



$g = h \circ f^{-1}$

quasigroupoid

$\Lambda^2_0$  $g = h \circ f^{-1}$

quasigroupoid

$\Lambda^2_2$  $f = g^{-1} \circ h$

minimal model structure — cofibrations fixed (to (normal) monos) {weak equiv} is minimal

Example. Cat ... cofibrations = injective on objects
weak equivalences = equivalences of categories

localization w.r.t. $(0 \longrightarrow 1) \longrightarrow (0 \xrightarrow{\cong} 1)$
$[1]$ $\quad I$

$[1] \xrightarrow{\text{mor}} \mathcal{C}$, $[1] \to I$, $I \xrightarrow{\text{iso}} \mathcal{C}$

local objects = groupoids

Remark. local objects are closed under homotopy limits

$\pi_1 \Omega X = \pi_2 X$

Example. Top / $sSet_{Kan}$

localization w.r.t. $f: S^{k+1} \longrightarrow D^{k+2} \rightsquigarrow \Lambda f: \underbrace{D^n \times S^{k+1} +_{S^{n-1} \times S^{k+1}} S^{n-1} \times D^{k+2}}_{S^{n+k+1}} \longrightarrow \overbrace{D^n \times D^{k+2}}^{D^{n+k+2}}$

local objects = spaces with trivial htpy groups $\pi_*$ for $* > k$
= homotopy $k$-types

$A^n +_{L^n A} L^n B \to B^n$
$A^n = \Delta^n \times A$
$L^n A = \partial\Delta^n \times A$

E.g. $k=1$ get 1-types with only $\pi_0, \pi_1$

$\Delta \to Cat$

$$\begin{array}{ccc} sSet & \underset{N}{\overset{\tau}{\rightleftarrows}} & Cat \\ S \uparrow \downarrow |\cdot| & & \\ Top & & \end{array}$$

$\Delta \to Top$

$0 \to 1 \to \cdots \to n$

$N\mathcal{C}_n = Cat([n], \mathcal{C}) \rightsquigarrow \tau\Delta^n = [n]$ + preserves colimits

$L_1 sSet \simeq_Q \underbrace{L_1 Cat}_{\text{groupoids}}$
$\|_Q$ $\quad$ homotopy 1-types
$L_1 Top$

$\Pi_1$ fundamental groupoid
ob: $x \in X$
mor: $[(I,0,1),(X,x,y)]$

The $k = \infty$ version of this is:

$Top \simeq_Q \infty\text{-}Gpd$ the so-called homotopy hypothesis

What is the localization ?

$X \underset{\sim_n}{\longrightarrow} L_n X$ obtained by attaching cells of dimension $\geq n+2$

$\llcorner$ homotopy $n$-type

$\Rightarrow$ $L_n X$ has the htpy groups of $X$ up to dimension $\underline{n}$ and trivial htpy groups above that

What are the local equivalences ?

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"', "\sim_n"] & L_n X \arrow[d, "\cong"] \\ Y \arrow[r] & L_n Y \end{tikzcd} \iff f \text{ induces iso on } \pi_* \text{ for } * \leq n.$$

$\Rightarrow$ the so-called <u>n-th</u> Postnikov section / stage of $X$

$$\begin{tikzcd} & \vdots \arrow[d] \\ X \arrow[r] \arrow[dr] & L_n X \arrow[d] \\ & L_{n-1} X \arrow[d] \\ & \vdots \end{tikzcd} \quad \Big\} \text{ Postnikov tower}$$

$$\begin{tikzcd} K(\pi_n X, n) \arrow[d] \\ L_n X \arrow[d] \\ L_{n-1} X \end{tikzcd}$$

Example. Homological localization   $g: X \to Y$ s.t. $g_*: E_* X \xrightarrow{\cong} E_* Y$

local equivalence = $E_*$-equivalences (iso in $E_*$)

Bousfield cardinality argument (simpler, I guess)

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \tilde{K} \arrow[r] \arrow[d, "\sim_E"] & X \arrow[d, "\sim_E"] \\ L \arrow[r] & \tilde{L} \arrow[r] & Y \end{tikzcd}$$

$K, L$ small $\Rightarrow \tilde{K}, \tilde{L}$ small

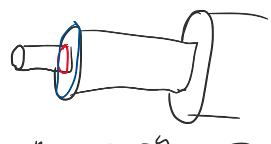
$$\Rightarrow \begin{tikzcd} \tilde{K} \arrow[r] \arrow[d, "\sim_E"] & X \arrow[d, "\sim_E"] \arrow[ddr, bend left, "\sim_E"] & \\ \tilde{L} \arrow[r] \arrow[drr, bend right] & X +_{\tilde{K}} \tilde{L} \arrow[dr] & \\ & & Y \end{tikzcd}$$

$\Rightarrow$ $E_*$-equivalences determined by small such

E.g. $E_* = H_*(-; \mathbb{Q})$   $g: X \to Y$ induces iso on $\pi_* \otimes \mathbb{Q}$
$\iff$ iso $H_*( ; \mathbb{Q})$
$\iff$ iso $H^*( ; \mathbb{Q})$

For simply connected spaces (e.g. 1-reduced simplicial sets?)

this should be equivalent to the localization w.r.t.

$n \geq 1$ or $2$: $S^n \rightarrowtail S^n_{\mathbb{Q}} =$  – – – mapping telescope

$\pi_n S^n = \mathbb{Z}$   $\pi_n S^n_{\mathbb{Q}} = \mathbb{Q}$

$I \times S^n$   $I \times S^n$   $I \times S^n$

$S^n \xrightarrow{2x} S^n$, $S^n \xrightarrow{3x} S^n$, $S^n \xrightarrow{4x} S^n$

$\mathbb{Q} = \operatorname{colim}(\mathbb{Z} \xrightarrow{2x} \mathbb{Z} \xrightarrow{3x} \mathbb{Z} \xrightarrow{4x} \cdots)$

$= \operatorname{hocolim}(S^n \xrightarrow{2x} S^n \xrightarrow{3x} S^n \xrightarrow{4x} \cdots)$

$S^1 \xrightarrow{z \mapsto z^2} S^1$

injectivity $\equiv$ $\pi_n \xrightarrow{\cong} \pi_n \otimes \mathbb{Q}$

$= \operatorname{hocolim}\left( S^n \xrightarrow{2\times} S^n \xrightarrow{3\times} S^n \xrightarrow{4\times} \cdots \right)$

$S^1 \xrightarrow{z \mapsto z^2} S^1$

$S^{n-1} S^1 \xrightarrow{S^{n-1}()} S^{n-1} S^1$

injectivity $\equiv$ $\pi_n \xrightarrow{\cong} \pi_n \otimes \mathbb{Q}$

$$Ho(L_{\mathbb{Q}} Top_{finite,\, 1\text{-conn}}) \simeq_{\mathbb{Q}} Ho(CDGA_{\mathbb{Q},\, 1\text{-conn}})^{op}$$

$X \longmapsto$ almost $C^*(X;\mathbb{Q})$

not commutative

rather $\Omega^*(X;\mathbb{Q})$ $A_{PL}$ Sullivan

polynomial forms

$\simeq_{\mathbb{Q}}$ $Ho(DGLA_{\mathbb{Q},\, 0\text{-conn}})$

$\pi_*$ with Whitehead product

$\simeq_{\mathbb{Q}}$ $Ho(CDGC_{\mathbb{Q},\, 1\text{-conn}})$

**Example.** Spectra

**Example.** (complete) Segal spaces

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$$\mathcal{N} = \quad 0 \xrightarrow{\circledcirc} 1 \xrightarrow{\circledcirc} 2 \xrightarrow{\circledcirc} \cdots \qquad \text{a } Top_*\text{-Cat}$$

$\mathcal{N}(0,1) = S^1 \in Top_*$

$\mathcal{N}(1,2) = S^1 \in Top_*$

$\vdots$

$$\mathcal{N}(i,j) = \underbrace{S^1 \wedge \cdots \wedge S^1}_{(j-i)\text{-times}} = S^{j-i}$$

$$\mathcal{N}(j,k) \wedge \mathcal{N}(i,j) \xrightarrow{\circ} \mathcal{N}(i,k)$$

$$S^{k-j} \wedge S^{j-i} \xrightarrow{\cong} S^{k-i}$$

$$(S^1 \wedge \cdots \wedge S^1) \wedge (S^1 \wedge \cdots \wedge S^1) \qquad S^1 \wedge \cdots \wedge S^1$$

$Sp = [\mathcal{N}, Top_*]$ presheaf category (enriched)

$$X: \quad \begin{array}{ccc} n & \longmapsto & X(n) \\ ()S^1 & \longrightarrow & () \; map(X(n), X(n+1)) \\ n+1 & \longmapsto & X(n+1) \end{array}$$

$$\frac{\dfrac{S^1 \longrightarrow map(X(n), X(n+1))}{S^1 \wedge X(n) \longrightarrow X(n+1)}}{X(n) \longrightarrow map(S^1, X(n+1))}$$

i.e. $\Sigma X(n) \longrightarrow X(n+1)$

i.e. $X(n) \xrightarrow{w_n} \Omega X(n+1)$

$$X(n) \longrightarrow \mathrm{map}(S^1, X(n+1)) \quad \text{i.e.} \quad X(n) \xrightarrow{w_n} \Omega X(n+1)$$

$[\mathcal{N}, Top_*]$ has a projective model structure = weak equiv, fibrations } ptwise

generated by $\partial D^k_+ \wedge \mathcal{N}_n \longrightarrow D^k_+ \wedge \mathcal{N}_n$

> $Top: \partial D^n \to D^n$
> $()_+ \downarrow \uparrow u$
> $Top_* = */Top: \partial D^n_+ \to D^n_+$
> Q: what does $\partial D^n \to D^n$ generate?

$\rightsquigarrow$ left Bousfield localization having local objects those with $w_n$ weak equivalences

$$w_n : X(n) \xrightarrow{\sim} \Omega X(n+1)$$

$$\mathrm{map}(\mathcal{N}_n, X) \xrightarrow{\sim} \mathrm{map}(S^1, \underbrace{\mathrm{map}(\mathcal{N}_{n+1}, X)}_{X(n+1)})$$

$$= \mathrm{map}(S^1 \wedge \mathcal{N}_{n+1}, X) \iff$$

$\iff X$ local w.r.t. $\mathcal{N}_n \longleftarrow S^1 \wedge \mathcal{N}_{n+1}$

LHS: $* \; \cdots \; * \;\; \underset{n}{S^0} \;\; \underset{n+1}{S^1} \;\; \underset{n+2}{S^2} \; \cdots$ — free diagram on $S^0$ sitting at $n \in \mathcal{N}$

$\quad\quad\quad\quad\quad\quad \uparrow\neq \quad \uparrow= \quad \uparrow=$

RHS: $* \; \cdots \; * \;\; * \;\; S^1 \wedge S^0 \;\; S^1 \wedge S^1$

"stable model structure" = left B.L. w.r.t. $S^1 \wedge \mathcal{N}_{n+1} \to \mathcal{N}_n$ $\forall n$

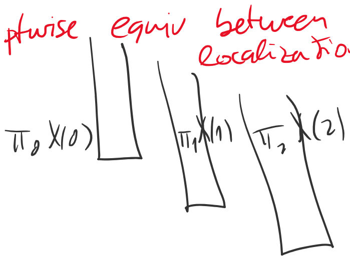
$X$ local $\quad X(0) \xrightarrow{\sim} \Omega X(1) \xrightarrow{\sim} \Omega^2 X(2) \xrightarrow{\sim} \Omega^3 X(3) \xrightarrow{\sim} \cdots$

delooping of $X(0)$

local equivalence = ptwise equiv between localizations

= stable equivalence = iso on $\pi_*$

$\pi_n X = \operatorname{colim}_{k \to \infty} \pi_{n+k} X(k)$




"infinite loop spaces" $\doteq$ hlgy theories

$\doteq$ stable htpy theory

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Ex. Segal spaces and complete S.S.

= weak internal categories in $\infty$-cat's

$X(0)$ space of objects $\quad X(0) \in sSet$

$X(1)$ space of mor's $\quad X(1) \in sSet$

$X(2)$ "composable" pairs of mor's $\quad X(2) \in sSet$

$X(n)$ n-tuples

$$X : \Delta^{op} \longrightarrow sSet$$
$$[n] \longmapsto X(n)$$

$$\begin{tikzcd}
X(2) \arrow[r, "d^2"] \arrow[d, "d^0"'] \arrow[dr, phantom, "\lrcorner", very near start] & X(1) \arrow[d, "d^0"] \\
X(1) \arrow[r, "d^1"'] & X(0)
\end{tikzcd}$$

$\Leftrightarrow$ $X(2)$ is exactly the space of comp'le pairs

Segal space:
$$\begin{tikzcd}
X(2) \arrow[r, "\sim"] \arrow[d, "d_1"'] & X(1) \times^{htpy}_{X(0)} X(1) \arrow[dl, dashed, "\text{composition}"] \\
X(1) &
\end{tikzcd}$$

localize w.r.t. $[1] +_{[0]} [1] \longrightarrow [2]$