Motivation

Literature:

- Hovey – Model categories
- ✗ Hirschhorn – Model categories and their localizations (reference book)
- Dwyer, Spalinski – Homotopy theories and model categories
- Quillen – Homotopical algebra (original)
- Goerss, Jardine – Simplicial homotopy theory (sSet-oriented) – quite categorical

Now some motivation:

- they are a formalism to do homotopy theory → "abstract homotopy theory" and compare
- examples:
  - topological spaces up to (weak) homotopy — htpy theory of CW-cxs
  - simplicial sets up to (weak) homotopy — htpy theory of Kan cxs
  - categories up to equivalence → ∞-categories
  - groupoids up to equivalence → ∞-groupoids
  - chain complexes (non-negatively graded, unbounded) up to homotopy equivalence / quasi-iso

Top + htpy classes of maps

$\cancel{X \overset{\text{htpy eq.}}{\cong} Y}$

w.h.e. $\rightsquigarrow$ iso

CW-cxs

h.e. $\equiv$ w.h.e.

$C \xrightarrow{\sim} D$ ⇝ iso

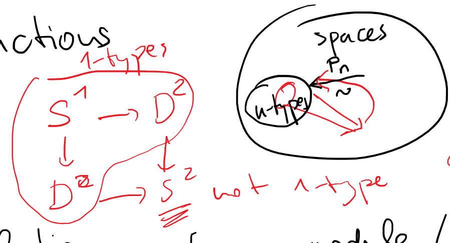
$H_*: Ch/q\text{-iso} \to Ab$ (iso)

- n-categories $\underset{?}{\simeq}$ homotopy n-types

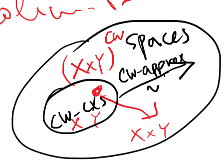
↑ spaces with homotopy groups concentrated in dimensions $0 \ldots n$ – not closed under categorical constructions

$\pi_i X = 0$ for $i > n$ →

colimits

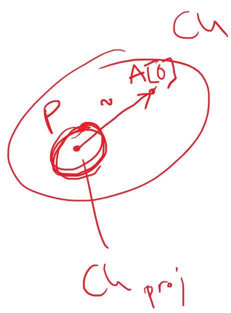


similarly:



- projective resolution of a module / chain cx
- free G-spaces $\subseteq$ G-spaces

⇒ homotopy theory of free G-spaces viewed as a structure on all G-spaces (categorically better behaved) that identifies any G-space with the corresponding free G-space



the corresponding free G-space
(reflection / coreflection)

Definition

A model category is a category M equipped with three classes of morphisms

- $\xrightarrow{\sim}$ • $\mathcal{W}$ ... weak equivalences — things to be inverted
- $\rightarrowtail$ • $\mathcal{C}$ ... cofibrations — things that behave well w.r.t. to constructions like colimits
- $\twoheadrightarrow$ • $\mathcal{F}$ .. fibrations — the same for limits
- $\overset{\sim}{\rightarrowtail}$ • $\mathcal{W}\cap\mathcal{C}$ ... trivial cofibrations
- $\overset{\sim}{\twoheadrightarrow}$ • $\mathcal{W}\cap\mathcal{F}$ ... trivial fibrations

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X +_A Y \end{tikzcd}$$

htpy inv. for cofibrations

$$\begin{tikzcd} S^1 \arrow[r, tail] \arrow[d, tail] & D^2 \simeq * \arrow[d] \\ * \simeq D^2 \arrow[r] & S^2 \not\simeq * \end{tikzcd}$$

Axioms: An important notion: lifting property

$g \,\square\, h$ means

$$\begin{tikzcd} \cdot \arrow[r, "\forall a"] \arrow[d, "g"'] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "\forall b"'] \arrow[ur, dashed, "\exists \ell"] & \cdot \end{tikzcd}$$

$\forall a \forall b$: square commutes $\Rightarrow \exists \ell$: triangles commute

g has LLP w.r.t. h

h has RLP w.r.t. g

! We have seen numerous examples:

surjective / injective / bijective mappings (in set)

injective modules & projective modules

serre/ Hurewicz fibrations

surj. zobr.

$$\begin{tikzcd} \emptyset \arrow[d] & X \arrow[d, "h"] \\ * & Y \end{tikzcd} \quad \square \quad \Longleftrightarrow \quad h \text{ je surj.}$$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

prvek X

$\downarrow h$

prvek Y

kdyby ex. jediné, byla by h bijekce

inj. zobr.

$$\begin{tikzcd} ** \arrow[r, "x_1, x_2"] \arrow[d] & X \arrow[d, "h"] \\ * \arrow[r, "y"'] \arrow[ur, dashed, "x"] & Y \end{tikzcd}$$

$\forall$ $h(x_1) = y = h(x_2)$

$\exists$ x : $x_1 = x = x_2$

$x_1 = x_2$

DÚ: inj. zobr. pomocí LLP

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd} ???$$

E injektivní modul $\overset{\text{def}}{\Longleftrightarrow}$ $A \rightarrowtail E$ ... $A \longrightarrow E$ $\overset{\text{Baer}}{\Downarrow}$ $I \longrightarrow E$

I injektivní modul (⟺)

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & E \\ B \arrow[ur, dashed] & \end{tikzcd} \iff \begin{tikzcd} A \arrow[r] \arrow[d, tail] & E \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & 0 \end{tikzcd} \Downarrow \iff \begin{tikzcd} I \arrow[r] \arrow[d, tail] & E \\ R \arrow[ur, dashed] & \end{tikzcd}$$

P projektivní modul ⟺

$$\begin{tikzcd} & A \arrow[d, two heads] \\ P \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \iff \begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, two heads] \\ P \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$ neumím ⟺

Serreova fibrace $p: X \to Y$

triv. c.f. →
$$\begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[d, hook, "i_n"] & X \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

p je S.f. ⟺ $\forall n: i_n \,\square\, p$

(Hurewiczova fibrace ... $D^n \rightsquigarrow$ lib. top. pr.)

$\mathcal{I} = \{ i_n \mid n \in \mathbb{N} \}$ ... p je S.f. ⟺ $\mathcal{I} \,\square\, p$

Pozn.
$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X \\ D^{n+1} \arrow[ur, dashed] & \end{tikzcd} \iff \pi_n X = 0$$

$\mathcal{G}, \mathcal{H}$ two classes of morphisms: $\mathcal{G} \,\square\, \mathcal{H} \iff \forall g \in \mathcal{G}\ \forall h \in \mathcal{H}\ g \,\square\, h$

$\Rightarrow \mathcal{G}^{\square} \supseteq \mathcal{H} \iff \mathcal{G} \,\square\, \mathcal{H} \iff \mathcal{G} \subseteq {}^{\square}\mathcal{H}$

$\mathcal{G}^{\square} = \{ h \mid \mathcal{G} \,\square\, h \}$ $\quad {}^{\square}\mathcal{H} = \{ g \mid g \,\square\, \mathcal{H} \}$

— factorization property:

$\mathcal{M} = \mathcal{H} \circ \mathcal{G}$ means: $\forall a\ \exists g \in \mathcal{G}, \exists h \in \mathcal{H}: a = h \circ g$

${}^{\square}(\mathcal{G}^{\square})$ ... uzávěr

$({}^{\square}\mathcal{H})^{\square}$ ... uzávěr

—

Now we are ready to state the axioms

$\mathcal{M} \supseteq \mathcal{W}$ ← subcat, $\mathcal{C}$, $\mathcal{F}$

MC1 (finite) small limits and colimits exist in $\mathcal{M}$

MC2 2-out-of-3 for $\mathcal{W}$: $\begin{tikzcd} \cdot \arrow[rr, "f \circ g"] \arrow[dr, "g"'] & & \cdot \\ & \cdot \arrow[ur, "f"'] & \end{tikzcd}$ 2 of these in $\mathcal{W}$ ⇒ so is 3rd

MC3 All $\mathcal{W}, \mathcal{C}, \mathcal{F}$ are closed under retracts

MC4 $\mathcal{C} \,\square\, (\mathcal{W} \cap \mathcal{F})$, $\quad (\mathcal{W} \cap \mathcal{C}) \,\square\, \mathcal{F}$

MC5 $\mathcal{M} = (\mathcal{W} \cap \mathcal{F}) \circ \mathcal{C}$, $\quad \mathcal{M} = \mathcal{F} \circ (\mathcal{W} \cap \mathcal{C})$

} $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ & $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$ form the so-called weak factorization systems

f is a retract of g ... in $\mathcal{M}^{\to}$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[rr, bend left, "1"] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

$| \ A \to B \to A$

$$\exists \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1"] & \cdot \arrow[r] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[rr, bend right, "1"'] & \cdot \arrow[r] & \cdot \end{tikzcd} \qquad \bigg| \qquad \begin{tikzcd} A \arrow[r] \arrow[rr, bend right, "1"'] & B \arrow[r] & A \end{tikzcd}$$

<u>Pozn.</u> To, že M je modelová kategorie, se dokazuje těžko.

$\Rightarrow$ mělo by se to vyplatit

$\Rightarrow$ příklady budou, <u>některé</u> podrobně

$$\mathcal{C} \ni i \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[ur, dashed, "\ell"] & \cdot \end{tikzcd} p \in \mathcal{F}$$

alespoň jedno z i, p je w.e. $\Rightarrow$ lift ℓ existuje
diagonála

<u>Pozn.</u> M s $\mathcal{W}, \mathcal{C}, \mathcal{F}$ je modelová kategorie
M^{op} s $\mathcal{W}^{op}, \mathcal{F}^{op}, \mathcal{C}^{op}$ je modelová kategorie

First properties

Retract argument:

$$f = g \circ h \quad \text{and} \quad f \,\square\, g \;\Rightarrow\; f \text{ is a retract of } h$$

Proof.
$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "1"'] \arrow[ur, dashed, "\ell"] & \cdot \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "f"'] & \cdot \arrow[r, "1"] \arrow[d, "h"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "\ell"'] \arrow[rr, bend right, "1"'] & \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

$\overset{\text{def}}{\equiv}$ f je retrakt h

∎

Lifting properties — v řeči Galoisovy konexe – $\mathcal{W}\cap\mathcal{C}$, $\mathcal{F}$ jsou uz.

- $\mathcal{W}\cap\mathcal{C} = {}^{\square}\mathcal{F}$, $(\mathcal{W}\cap\mathcal{C})^{\square} = \mathcal{F}$
- $\mathcal{C} = {}^{\square}(\mathcal{W}\cap\mathcal{F})$, $\mathcal{C}^{\square} = \mathcal{W}\cap\mathcal{F}$

 └ první bod pro $\mathcal{M}^{op}$

⇒ $\mathcal{W}, \mathcal{C}$ určují $\mathcal{F}$
$\mathcal{W}, \mathcal{F}$ určují $\mathcal{C}$
$\mathcal{C}, \mathcal{F}$ určují $\mathcal{W}$

$\mathcal{C}^{\square} = \mathcal{W}\cap\mathcal{F}$ ${}^{\square}\mathcal{F} = \mathcal{W}\cap\mathcal{C}$

Důkaz. $\mathcal{W}\cap\mathcal{C} \subseteq {}^{\square}\mathcal{F}$ axiom MC4

zbývá $\mathcal{W}\cap\mathcal{C} \supseteq {}^{\square}\mathcal{F} \ni f$ .... tj. $f \,\square\, \mathcal{F}$

použijeme MC5
$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "h"', "\sim"] & & \cdot \\ & \cdot \arrow[ur, "g"'] & \end{tikzcd}$$

$f = g \circ h$ $\quad f \,\square\, \mathcal{F} \ni g \overset{RA}{\Rightarrow} f$ retrakt h

$\Rightarrow h \in \mathcal{W}\cap\mathcal{C}$, f retrakt $h \overset{MC3}{\Rightarrow} f \in \mathcal{W}\cap\mathcal{C}$. □

Stability properties of the classes ${}^{\square}\mathcal{H}$, $\mathcal{G}^{\square}$ ⇒ in particular for $\mathcal{C}, \mathcal{W}\cap\mathcal{C}$, $\mathcal{F}, \mathcal{W}\cap\mathcal{F}$

→ ${}^{\square}\mathcal{H}$ je uz. na různé konstrukce
– koprodukty, pushouty, transf. kompozice, retrakty

→ příště

Meta-definition of the homotopy category, universal property

Cofibrant and fibrant replacement

Homotopy relation on maps, homotopy category

DIFFICULT ?

$\boxed{\text{Examples}}$ – DIFFICULT !