First properties – continued

Stability properties of the classes ${}^\square\mathcal{H}$, $\mathcal{G}^\square$ $\Rightarrow$ in particular for $\mathcal{C}, \mathcal{W}\cap\mathcal{C}$, $\mathcal{F}, \mathcal{W}\cap\mathcal{F}$

$\mathcal{C} = {}^\square(\mathcal{W}\cap\mathcal{F})$

Theorem. The class ${}^\square\mathcal{H}$ is closed under

- coproducts
- pushouts
- transfinite compositions
- retracts

$\longleftarrow$ certain colimit constructions

closed in the sense of Galois conn

Rk. If $\mathcal{I}$ is a set of small objects, the class ${}^\square(\mathcal{I}^\square)$ coincides with the closure under these constructions (small object argument – to be proved later) $\leadsto$ this is a complete list (roughly).

Proof.

- $\{A_s \xrightarrow{f_s} B_s\}_{s\in S} \,\square\, \mathcal{H} \overset{?}{\Rightarrow} \left(\sum_s A_s \xrightarrow{\sum f_s} \sum_s B_s\right) \square\, \mathcal{H}$

A_1+A_2, $\sum A_s$ | $A_1\times A_2$, $\prod A_s$

$$\begin{tikzcd} \sum A_s \arrow[r] \arrow[d] & X \arrow[d, "h"] \\ \sum B_s \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \equiv \quad \forall s\in S: \begin{tikzcd} A_s \arrow[r] \arrow[d, "f_s"'] & X \arrow[d, "h"] \\ B_s \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

vlevo komutuje $\Leftrightarrow$ $\forall s$: vpravo komutuje

-

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad f \,\square\, \mathcal{H} \overset{?}{\Rightarrow} f' \,\square\, \mathcal{H}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "h"] \\ B \arrow[r] & B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "h"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[urr] & B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

- $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots$

  A_λ; indexed by ordinals $\lambda < \alpha$ $\longleftarrow$ diagram $\alpha \to \mathcal{M}$, $\{0<1<\cdots\} = \{\lambda \mid \lambda<\alpha\}$

  for $\lambda<\mu<\alpha$: $A_\lambda \to A_\mu$, closed under comp'n

  - $f_\lambda : A_\lambda \to A_{\lambda+1}$
  - μ limit: $\operatorname{colim}_{\lambda<\mu} A_\lambda \xrightarrow{\cong} A_\mu$

  $A_0 \to A_1 \to \cdots$ $\operatorname{colim}_{\lambda<\omega_0} A_\lambda \to A_{\omega_0}$

  $\rightarrow$ chain; it is said to be smooth if the maps in the second point are isomorphisms

  component of the univ. cone

if the maps in the second point are isomorphisms

component of the univ. cone

$\to$ there is an induced map $A_0 \longrightarrow \operatorname{colim}_{\lambda<\alpha} A_\lambda$ called the (transfinite) composition of the chain

<u>Claim:</u> in a smooth chain: $f_\lambda \in {}^{\square}\mathcal{H} \overset{?}{\Rightarrow}$ the composition $\left(A_0 \longrightarrow \operatorname{colim}_{\lambda<\alpha} A_\lambda\right) \in {}^{\square}\mathcal{H}$

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & X \arrow[d, "h"] \\ \operatorname{colim} A_\lambda \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A_0 \arrow[rr] \arrow[d, "f_0"'] & & X \arrow[d] \\ A_1 \arrow[r] \arrow[urr, dashed] & \operatorname{colim} A_\lambda \arrow[r] & Y \end{tikzcd}$$

at isolated steps

next step:

$$\begin{tikzcd} A_1 \arrow[rr] \arrow[d, "f_1"'] & & X \arrow[d] \\ A_2 \arrow[r] \arrow[urr, dashed] & \operatorname{colim} A_\lambda \arrow[r] & Y \end{tikzcd}$$

$$\begin{tikzcd} A_0 \arrow[d] \arrow[rrr] & & & X \arrow[dddd] \\ A_1 \arrow[d] \arrow[urrr] & & & \\ \vdots & & & \\ \operatorname{colim}_{\lambda<\mu} A_\lambda \arrow[d, "\cong"] \arrow[uuurrr, dashed] & & & \\ A_\mu \arrow[r] \arrow[uuuurrr, dashed] & \operatorname{colim} A_\lambda \arrow[rr] & & Y \end{tikzcd}$$

(would be enough $\to$ to assume that this belongs to ${}^{\square}\mathcal{H}$)

$\alpha = 3 = \{0 \to 1 \to 2\}$

$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$

$f_0, f_1 \in {}^{\square}\mathcal{H} \Rightarrow$
$\Rightarrow f_1 \circ f_0 \in {}^{\square}\mathcal{H}$

- retracts similar, HW.

$\square$

$\Rightarrow$ <u>Corollary.</u> the classes $\mathcal{C}$ of cofibrations and wne of trivial cofibrations are closed under these constructions.

<u>Remark.</u> These construction will be important in SOA, that is the main in proving that "anything" is a model cat.

gives functorial cof. repl.

Intuition from CW-complexes:

- $$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d, "i_n"'] & A \arrow[d] \\ D^n \arrow[r] & B \end{tikzcd}$$ $\longleftarrow$ B is obtained from A by attaching a n-dimensional cell (of shape i_n --- various shapes of cells)

- $$\begin{tikzcd} \sum_s \partial D^{n_s} \arrow[r] \arrow[d] & A \arrow[d] \\ \sum_s D^{n_s} \arrow[r] & B \end{tikzcd}$$ $\longleftarrow$ B is obtained from A by attaching multiple cells at one time

- as in the second point

  $A_0 \to A_1 \to \cdots \operatorname{colim} A_\lambda = B$ $\longleftarrow$ B is obtained

$\Rightarrow$ no need to attach cells of increasing dim's. ⌉ inductively by attaching cells

- retracts are weird

In Top, $I = \{ i_n : \partial D^n \to D^n \}$ will "generate" cofibrations in the sense that $\mathcal{C} = {}^{\square}(I^{\square}) \Rightarrow \mathcal{C}$ contains all "relative cell complexes". The cell complexes are those built from the empty set;

Definition. An object $A \in \mathcal{M}$ is said to be cofibrant if the unique map from the initial object 0 to A is a cofibration. $0 \rightarrowtail A$

We denote by $\mathcal{M}_c$ the full subcat of $\mathcal{M}$ on cofibrant objects.

terminal

$\mathcal{M}_f$ fibrant ... $A \twoheadrightarrow 1$

$\mathcal{M}_{cf} = \mathcal{M}_c \cap \mathcal{M}_f$

Remark. By the above, all CW-complexes (and more general cell complexes) in Top will be cofibrant.

Definition. A cofibrant replacement of an object $X \in \mathcal{M}$ is a weak equivalence $A \xrightarrow{\sim} X$ from a cofibrant A,

Diagrammatically:

$$0 \rightarrowtail A \xrightarrow{\sim} X.$$

In particular, we get one from the factorization axiom:

$$0 \rightarrowtail A \overset{\sim}{\twoheadrightarrow} X$$

unique map

better, a trivial fibration

Ex. Any space is w.h.e. to a CW-complex.

Ex. Any chain complex is q-iso ch. cx of proj. mod.

$P \xrightarrow{\sim} A[0]$

$\llcorner$ proj. res. A

Corollary. $A, B \in \mathcal{M}_c \Rightarrow A \rightarrowtail \underset{\mathcal{M}_c}{\underset{\cap}{A+B}} \leftarrowtail B.$

Proof. Coproduct is a pushout:

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] & A \arrow[d, tail] \\
B \arrow[r, tail] & A+B \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

the indicated maps are cofibrations and so is the composition $0 \to A+B$. $\square$

We have: $\mathcal{M}$ with $\mathcal{W}, \mathcal{C}, \mathcal{F}$ a model cat

$\Rightarrow \mathcal{M}^{op}$ with $\mathcal{W}^{op}, \mathcal{F}^{op}, \mathcal{C}^{op}$ a model cat (last time)

$\mathcal{M}_s$ model cat's $\Rightarrow \prod_s \mathcal{M}_s$ a model cat (everything componentwise) $\mathcal{M}_1 \times \mathcal{M}_2$

Comma categories:

$A \in \mathcal{M} \Rightarrow A/\mathcal{M}$ has
- objects $A \xrightarrow{x} X$
- morphisms
$$\begin{tikzcd}
& X \arrow[d, "f"] \\
A \arrow[ur, "x"] \arrow[r] & Y
\end{tikzcd}$$

EX. Top model cat $\Rightarrow */\mathrm{Top} = \mathrm{Top}_*$ also model cat

- morphisms
$$\begin{tikzcd} & X \arrow[d, "f"] \\ A \arrow[ur, "x"] \arrow[r, "y"'] & Y \end{tikzcd}$$

$\Rightarrow$ $*/Top = Top_*$ also model cat

this inherits a model structure:

f is a w.e. / cof / fib in A/M $\Leftrightarrow$ it is one in M

HW. Check the axioms (colimits are a bit tricky, may ignore).

What is a cofibrant object in A/M?

$$\begin{tikzcd} & A \arrow[d, "b"] \\ A \arrow[ur, "1"] \arrow[r, "b"'] & B \end{tikzcd}$$

$\leftarrow$ the initial object in A/M

$\leftrightarrow$ a cofibration with domain A

$$\begin{tikzcd} & X \\ A \arrow[ur] \arrow[dr] & \\ & Y \end{tikzcd}$$

coproduct

$$\begin{tikzcd} & X \arrow[dr] & \\ A \arrow[ur] \arrow[dr] \arrow[rr] & & X +_A Y \\ & Y \arrow[ur] & \end{tikzcd}$$

A fibrant object in A/M:

$$\begin{tikzcd} & B \arrow[d, two heads] \\ A \arrow[ur, "b"] \arrow[r] & 1 \end{tikzcd}$$

$\leftrightarrow$ any map with B fibrant

$\leftarrow$ the terminal object in A/M

Remark. A cofibrant replacement in A/M:

$$\begin{tikzcd} & B \arrow[d, "\sim"] \\ A \arrow[ur, tail] \arrow[r] & X \end{tikzcd}$$

B $\leftarrow$ cof. obj.

$\sim$ w.e.

X $\leftarrow$ object

a cof repl. of $A \to X$ is $\equiv$ a factorization $A \rightarrowtail B \xrightarrow{\sim} X$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[dr] & X^{cw} \arrow[d, "\sim"] \\ & X \end{tikzcd}$$

cof. repl. in A/Top

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & BG \end{tikzcd}$$

hl. G-bandl nad X

$$\begin{tikzcd} G \subset P \arrow[r] \arrow[d] & Q \supset G \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

The homotopy category

We are interested in the localization $\mathcal{W}^{-1}\mathcal{M}$

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[dr, "F"'] & \mathcal{W}^{-1}\mathcal{M} \arrow[d, dashed, "\exists!"] \\ & \mathcal{K} \end{tikzcd}$$

assuming that F takes $\mathcal{W}$ to iso's
(also has a weak property
- commutativity up to nat. iso
- uniqueness up to nat iso

→ non-evil notion)

Remark. One way construct $\mathcal{W}^{-1}\mathcal{M}$ if $\mathcal{M}$ is small (morphisms are formal compositions of morphisms in $\mathcal{M}$ and formal inverses of morphisms from $\mathcal{W}$, modulo relations)
"Grothendieck universes" — can always assume $\mathcal{M}$ to be small in some enlargement of Set

$\mathcal{W}^{-1}\mathcal{M}$ will be some homotopy category → will study now the homotopy relation in abstract $\mathcal{M}$:

- htpy = map from the cylinder ; also need cst_f htpy $I\times X \xrightarrow{p} X \xrightarrow{f} Y$

↓ $h: f \sim g$; ↓ two ends $X \xrightarrow{i_-} I\times X \xleftarrow{i_+} X$

Definition. A cylinder for $X \in \mathcal{M}$ is a diagram

$$\begin{tikzcd} X \arrow[d, "i_-"'] \arrow[dr, "1"] & \\ \mathrm{Cyl}X \arrow[r, "p"] & X \\ X \arrow[u, "i_+"] \arrow[ur, "1"'] & \end{tikzcd} \equiv \begin{tikzcd} X+X \arrow[r, tail, "{[i_-,i_+] = i}"] \arrow[rr, bend right, "\nabla = {[1,1]}"'] & \mathrm{Cyl}X \arrow[r, "p"', "\sim"] & X \end{tikzcd}$$

in which
- i is a cofibration
- p is a weak equivalence

cof repl ∇ in $X+X/\mathcal{M}$

$X \in \mathrm{Top}$ not a cell complex $\Rightarrow I\times X$ not a cylinder

Lemma. The maps $i_\pm$ are weak equivalences.
If X is cofibrant then $i_\pm$ are trivial cofibrations.

Proof.

$$\begin{tikzcd} X \arrow[d, "\mathrm{in}_\pm"'] \arrow[dr, "i_\pm", "\sim"'] \arrow[drr, bend left, "1"] & & \\ X+X \arrow[r, tail, "i"'] & \mathrm{Cyl}X \arrow[r, "\sim", "p"'] & X \end{tikzcd}$$

□

Before going into homotopies, we prove an essential lemma:

Ken Brown's lemma. Any w.e. between cofibrant objects factors as

$A \overset{\sim}{\rightarrowtail} C \xrightarrow{p} B$

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[rr, bend left, "f"] & C \arrow[r, "\sim"] \arrow[l, bend right, "\sim"'] & B \end{tikzcd}$$

factors as

$$f : A \overset{\sim}{\underset{i}{\rightarrowtail}} C \overset{p}{\twoheadrightarrow} B$$

with j ← a right inverse of p, $p \circ j = 1$

$$A \xrightarrow[i]{\sim} C \xrightarrow[p]{\sim} B, \quad f = p \circ i$$

i.e.

- $f = p \circ i$
- $i \in \mathcal{W} \cap \mathcal{C}$
- p has a right inverse $j \in \mathcal{W} \cap \mathcal{C}$

Corollary. If $F : \mathcal{M}_c \to \mathcal{X}$ is a functor that takes trivial cofibrations to iso's then it takes all w.e.'s (between cofibrant objects) to iso's. ← related to $\mathcal{W}^{-1}\mathcal{M}_c \xrightarrow{\simeq} \mathcal{W}^{-1}\mathcal{M}$

Pf of Cor. Fi iso, Fj iso,

$$\begin{tikzcd} & FC \arrow[dr, "Fp"] & \\ FB \arrow[ur, "Fj", "\cong"'] \arrow[rr, "1", "\cong"'] & & FB \end{tikzcd}$$

$\Rightarrow Fp$ iso $\Rightarrow Ff = Fp \circ Fi$ iso. □

Pf of Lemma. $A \xrightarrow{f} B$, C is the mapping cylinder

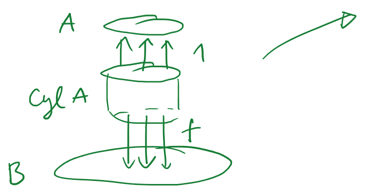
$C = \mathrm{Cyl}\,A +_f B$? This is equivalently a pushout

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "i_+", "\sim"'] & B \arrow[d, "j", "\sim"'] \\ A \arrow[r, tail, "i_-"] \arrow[rr, bend right, "i"'] & \mathrm{Cyl}\,A \arrow[r] & \mathrm{Cyl}\,A +_f B \end{tikzcd}$$

or

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "in_+"] & B \arrow[d, "in_+"] & \\ A+A \arrow[r, "1+f"] \arrow[d, "i"] & A+B \arrow[d] & A \arrow[l, "in_-"'] \arrow[dl, "i"] \\ \mathrm{Cyl}\,A \arrow[r] \arrow[dr, "p"] & \mathrm{Cyl}\,A +_f B \arrow[d, "p"] \arrow[d, bend left, "{[f,1]}"] & \\ & A & \end{tikzcd}$$

□



Definition. A *left homotopy* from f_- to f_+ w.r.t. a cylinder $\mathrm{Cyl}\,A$ is a map $h : \mathrm{Cyl}\,A \to X$. A left htpy = w.r.t. any cylinder.

$$\begin{tikzcd} \mathrm{Cyl}\,A \arrow[r, "h"] & X \\ A+A \arrow[u, "i"] \arrow[ur, "{[f_-, f_+]}"'] & \end{tikzcd}$$

Proposition. Suppose that A is cofibrant. Then left htpy is an equivalence relation on $\mathcal{M}(A, X)$ that respects composition on the left: $\mathcal{M}(A,X) \xrightarrow{g \circ} \mathcal{M}(A,Y)$, $f_- \mapsto g \circ f_-$, $f_+ \mapsto g \circ f_+$ ($\simeq \Rightarrow \simeq$)

Proof. • $f \sim f$ via the cst htpy

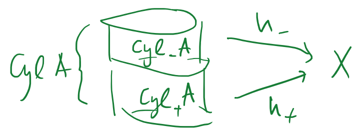
$$\mathrm{Cyl}\,A \xrightarrow{p} A \xrightarrow{f} X$$

- $f_- \sim f_+ \Rightarrow f_+ \sim f_-$ by swapping $i_+ \leftrightarrow i_-$ (a different cylinder ?)
- $f_- \sim f_0 \sim f_+ \Rightarrow f_- \sim f_+$ by "composing homotopies" ... rather gluing, yet another cylinder

$A \xrightarrow{i_-} \mathrm{Cyl}\,A \quad p_-$

$\mathrm{Cyl}_-A \xrightarrow{h_-} X$

$$\begin{tikzcd}
A \arrow[dr, "i_-"] & & & \\
& Cyl_-A \arrow[dr, "\sim"] \arrow[rrr, bend left, "p_-", "\sim"'] & & & \\
A \arrow[ur, "i_+", "\sim"'] \arrow[dr, "i_-"', "\sim"] & & Cyl\, A \arrow[r, dashed, "p", "\sim"'] & A \\
& Cyl_+A \arrow[ur, "\sim"'] \arrow[urr, bend right, "p_+"', "\sim"] & & \\
A \arrow[ur, "i_+"'] & & &
\end{tikzcd}$$



$$\begin{tikzcd}
& Cyl_-A \arrow[dr] \arrow[drr, bend left, "h_-"] & & \\
A \arrow[ur, "i_+"] \arrow[dr, "i_-"'] & & Cyl\,A \arrow[r, dashed, "h"] & X \\
& Cyl_+A \arrow[ur] \arrow[urr, bend right, "h_+"'] & &
\end{tikzcd}$$

- $h: f_- \sim f_+ \Rightarrow g\circ h : g\circ f_- \sim g\circ f_+$

$$\begin{tikzcd} Cyl\,A \arrow[r, "h"] \arrow[rr, bend right, "g\circ h"'] & X \arrow[r, "g"] & Y \end{tikzcd}$$

Dually: – path object $\begin{tikzcd} X \arrow[r, "d", "\sim"'] \arrow[rr, bend right, "\Delta"'] & Path\, X \arrow[r, "q"] & X\times X \end{tikzcd}$ $\square$

$V\ Top$: $X \to X^I \to X\times X$ (ev_0, ev_1)

– right homotopy – for X fibrant eq. rel. + preserved by $\mathcal{M}(B,X) \xrightarrow{f^*} \mathcal{M}(A,X)$

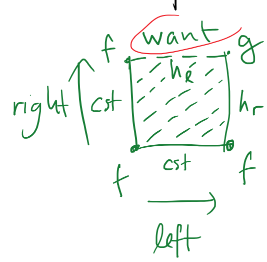
$A \to Path\ X$

Theorem. If A is cofibrant and X is fibrant then left and right homotopy relations coincide. $f_+\sim f_- \Rightarrow g\circ f_+ \sim g\circ f_-$

$\Rightarrow$ on $\mathcal{M}_{cf}$ – of cof. & fib objects – get a congruence $\Rightarrow Ho(\mathcal{M}_{cf}) = \mathcal{M}_{cf}/\sim$ quotient cat $\Rightarrow f_+\circ g \sim f_-\circ g$

Proof. Let $h_r: A \to Path\,X$ be a right htpy $h_r: f\sim g$ and $Cyl\,A$ any cylinder object. We will construct a left htpy $h_\ell: Cyl\,A \to X$, $h_\ell: f\sim g$:

Idea: a double htpy $Cyl\ A \to Path\ X$



$$A+A \xrightarrow{f+1} X+A \xrightarrow{[j,h_r]} Path\ X$$

$$\begin{tikzcd}
A+A \arrow[r, "cst_f + h_r"] \arrow[d, "i"'] & Path\ X \arrow[r, "q_+"] \arrow[d, "q_-", "\sim"'] & X \\
Cyl\ A \arrow[r, "cst_f"'] \arrow[ur, dashed, "H"] & X &
\end{tikzcd}$$

$h_\ell = q_+\circ H : Cyl\ A \to X$

důležité! h i_- je triv. kof.

$Cyl\ A \xrightarrow{p} A \xrightarrow{f} X$ $\square$

Remark. We proved, in fact, that htpy rel'ns w.r.t. any cylinder / path space coincide.

Theorem ("Whitehead"). $A, X \in \mathcal{M}_{cf}$, $f: A\to X$; then f is a w.e. $\iff$ f is a h.e.

homotopy equivalence = iso in $Ho(\mathcal{M}_{cf})$

Corollary. The canonical (projection) functor $\mathcal{M}_{cf} \to Ho(\mathcal{M}_{cf})$

presents $Ho(M_{cf})$ as the localization $W^{-1}M_{cf}$.
In addition, the class W is saturated in M_{cf} in that it consists of all maps that get inverted in $W^{-1}M_{cf}$.

Pf of Cor.

$$\begin{tikzcd} & Ho(M_{cf}) = M_{cf}/\sim \arrow[d, dashed] \\ M_{cf} \arrow[ur] \arrow[r, "F"'] & K \end{tikzcd}$$

$\leftarrow$ has univ. prop.

We need to show that $F(W) \subseteq Iso(K) \implies f_- \sim f_+ \implies Ff_- = Ff_+$.

$$\begin{tikzcd} FA \arrow[d, "Fi_-"'] \arrow[dr, "1"] & \\ F\,Cyl\,A \arrow[r, "Fp", "\cong"'] & FA \\ FA \arrow[u, "Fi_+"] \arrow[ur, "1"'] & \end{tikzcd}$$

Fi_-, Fi_+ right inverses of an iso Fp

$\implies Fi_- = Fi_+ \implies$

$$\begin{tikzcd} FA \arrow[d, shift right, "Fi_-"'] \arrow[d, shift left, "Fi_+"] \arrow[dr, bend left, "Ff_+"] \arrow[dr, "Ff_-"'] & \\ F\,Cyl\,A \arrow[r, "Fh"'] & FX \end{tikzcd}$$

∎

Pf of thm. • $f : X \xrightarrow{\sim} Y$ need to find a htpy inverse.
Factor $f : X \underset{i}{\overset{\sim}{\rightarrowtail}} Z \underset{p}{\overset{\sim}{\twoheadrightarrow}} Y$, enough for i & p ... dual, will do for i
$Z \in M_{cf}$

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, tail, "i"', "\sim"] & X \arrow[d, two heads] \\ Z \arrow[r] \arrow[ur, dashed, "g"] & 1 \end{tikzcd}$$

$g \circ i = 1$
$i \circ g \overset{?}{\simeq} 1$

$$\begin{tikzcd} X \arrow[r, "cst_i"] \arrow[d, tail, "i"', "\sim"] & Path\,Z \arrow[d] \\ Z \arrow[r, "(i\circ g, 1)"'] \arrow[ur, dashed] & Z\times Z \end{tikzcd}$$

upon composing with i
$\underbrace{i\circ g\circ i}_{1} = 1 \circ i$

• the opposite direction is a bit tricky.

$f : X \to Y$ h.e., factor $f : X \underset{i}{\overset{\sim}{\rightarrowtail}} Z \underset{p}{\twoheadrightarrow} Y$ and we need that it is a w.e.
i is a h.e. $\implies$ p is a h.e.
s' a htpy inverse

$$\begin{tikzcd} Y \arrow[r, "s'"] \arrow[d, "i_-"'] & Z \arrow[d, "p"] \\ Cyl\,Y \arrow[r, "h"'] \arrow[ur, dashed, "\bar h"] & Y \end{tikzcd}$$

"homotopy lifting property" $\rightsquigarrow$ $\bar h : s' \sim s$ such that s is a section, $p\circ s = 1$

↳ $h : p\circ s' \sim 1$ ... s' is a "homotopy section"

Still we have $s\circ p \sim 1 \implies s\circ p$ is a w.e.:

$$\begin{tikzcd} Z \arrow[d, "i_-"'] \arrow[dr, bend left, "s\circ p"] & \\ Cyl\,Z \arrow[r, "htpy"] & Z \\ Z \arrow[u, "i_+", "\sim"'] \arrow[ur, "1"'] & \end{tikzcd}$$

General fact: $f_- \sim f_+$ then f_- w.e. $\Leftrightarrow$ f_+ w.e.

Finally, p is a retract of $s\circ p$, thus a w.e.:

$$\begin{tikzcd} Z \arrow[r, "1"] \arrow[d, "p"'] & Z \arrow[r, "1"] \arrow[d, "s\circ p"] & Z \arrow[d, "p"] \\ Y \arrow[r, "s"'] & Z \arrow[r, "p"'] & Y \end{tikzcd}$$

∎

Construction. Choose, for each $M \in \mathcal{M}$ a cofibrant and fibrant repl.

$$M^{cf} \xleftarrow{\sim} M^c \xrightarrow{\sim} M$$

Define $\quad Ho(\mathcal{M})\,(M,N) = Ho(\mathcal{M}_{cf})\,(M^{cf}, N^{cf})$

Theorem. There is a canonical functor $\mathcal{M} \to Ho(\mathcal{M})$ that displays $Ho(\mathcal{M})$ as the localization $\mathcal{W}^{-1}\mathcal{M}$. In addition, the class $\mathcal{W}$ is saturated.

Proof.

$$\begin{tikzcd}
\mathcal{M}_{cf} \arrow[r] \arrow[d, hook] & Ho(\mathcal{M}_{cf}) \arrow[d, hook] \arrow[ddr, dashed, bend left, "\bar{F}\ \exists!"] & \\
\mathcal{M} \arrow[r] \arrow[drr, bend right, "F"'] & Ho(\mathcal{M}) \arrow[dr, dashed, "G\ \exists!"] & \\
 & & \mathcal{K}
\end{tikzcd}$$

$$\begin{tikzcd}
M^c \arrow[r, dashed, "\exists"] \arrow[d] & N^c \arrow[d, "\sim"] \\
M \arrow[r] & N
\end{tikzcd}$$

+ uniqueness up to left htpy

dually the same for $(\,)^f$

$\Rightarrow$ well defined functor to $Ho(\mathcal{M})$

$$\begin{array}{ccc}
G: Ho(\mathcal{M})(M,N) & \longrightarrow & \mathcal{K}(GM, GN) \\
\| & & \cong \\
\mathcal{M}(M^{cf}, N^{cf})/\sim & \xrightarrow{\ \bar{F}\ } & \mathcal{K}(GM^{cf}, GN^{cf})
\end{array}$$

$\square$