# The homotopy category continued

**Recall.**

- Cylinder $A+A \rightarrowtail \mathrm{Cyl}\,A \xrightarrow{\sim} A$
- Homotopy (left)

$$\begin{tikzcd}
A+A \arrow[r, "{[f_-,f_+]}"] \arrow[d, tail] & X \\
\mathrm{Cyl}\,A \arrow[ru, "h"'] &
\end{tikzcd}$$

- A cofibrant, X fibrant ⟹ on $\mathcal{M}(A,X)$ left and right htpy agree $\mathcal{M}(A,X)/\sim \; = [A,X]$ htpy classes

by using this we require $A \in \mathcal{M}_c$, $X \in \mathcal{M}_f$

- $\mathrm{Ho}(\mathcal{M}_{cf}) \overset{\mathrm{def}}{=} \mathcal{M}_{cf}/\sim$ ... cofibrant & fibrant objects, htpy classes of maps $\mathrm{Ho}(\mathcal{M}_{cf})(A,X) = [A,X]$

quotient category $\mathcal{M}_{cf} \to \mathrm{Ho}(\mathcal{M}_{cf})$ has a universal property

**Proposition** A w.e. $f : A \xrightarrow{\sim} B$ induces iso $f^* : [B,X] \longrightarrow [A,X]$

A w.e. $f : X \xrightarrow{\sim} Y$ induces iso $f_* : [A,X] \longrightarrow [A,Y]$.

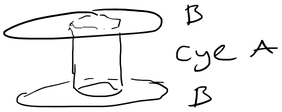
$A, B \in \mathcal{M}_c$
$X, Y \in \mathcal{M}_f$

**Proof.** By Brown's lemma (and duality) ⇝ enough for $j : A \overset{\sim}{\rightarrowtail} B$

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, tail, "j"', "\sim"] & X \arrow[d] \\
B \arrow[r] \arrow[ru, dashed, "g'"] & 1
\end{tikzcd}$$

$\Rightarrow [B,X] \xrightarrow{j^*} [A,X]$ surjective (already on maps)

Construct the "double mapping cylinder" $\mathrm{Cyl}(j,j)$:



$$\begin{tikzcd}
A+A \arrow[r, tail, "j+j", "\sim"'] \arrow[d, tail] & B+B \arrow[d, tail] \arrow[rrd, bend left, "\nabla"] & & \\
\mathrm{Cyl}\,A \arrow[r, tail, "\sim"] \arrow[rd, bend right, "p"', "\sim"] & \mathrm{Cyl}(i,i) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, tail] \arrow[rr, bend right] & \mathrm{Cyl}\,B \arrow[r, "\sim"] & B \\
& A \arrow[rru, bend right, "j"', "\sim"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Cyl}(j,j) \arrow[r, "{[g'_-, g'_+, h]}"] \arrow[d, tail, "\sim"] & X \\
\mathrm{Cyl}\,B \arrow[ru, dashed, "k"'] &
\end{tikzcd}$$

$\square$

**Theorem ("Whitehead").** $A, X \in \mathcal{M}_{cf}$, $f : A \to X$; then

f is a w.e. ⟺ f. is a h.e.
└ homotopy equivalence
= iso in $\mathrm{Ho}(\mathcal{M}_{cf})$

**Corollary.** The canonical (projection) functor

$\mathcal{M}_{cf} \longrightarrow \mathrm{Ho}(\mathcal{M}_{cf})$

presents $\mathrm{Ho}(\mathcal{M}_{cf})$ as the localization $\mathcal{W}^{-1}\mathcal{M}_{cf}$.

In addition, the class $\mathcal{W}$ is saturated in $\mathcal{M}_{cf}$ in that it consists of all maps that get inverted in $\mathcal{W}^{-1}\mathcal{M}_{cf}$.

**Pf of Cor.**

$$\begin{tikzcd}
& \mathrm{Ho}(\mathcal{M}_{cf}) = \mathcal{M}_{cf}/\sim \arrow[d, dashed] \\
\mathcal{M}_{cf} \arrow[ru] \arrow[r, "F"'] & K
\end{tikzcd}$$

⟵ has univ. prop.

We need to show that $F(\mathcal{W}) \subseteq \mathrm{Iso}(K) \Longrightarrow f_- \sim f_+ \Rightarrow Ff_- = Ff_+$.

We need to show that $F(\mathcal{W}) \subseteq Iso(\mathcal{K}) \Rightarrow f_- \sim f_+ \Rightarrow Ff_- = Ff_+$.

$$\begin{tikzcd}
FA \arrow[d, "Fi_-"'] \arrow[dr, "1"] & \\
F\,Cyl\,A \arrow[r, "Fp", "\cong"'] & FA \\
FA \arrow[u, "Fi_+"] \arrow[ur, "1"'] &
\end{tikzcd}$$

$Fi_-, F_+$ right inverses of an iso $Fp$

$\Rightarrow Fi_- = Fi_+ \Rightarrow$
$$\begin{tikzcd}
FA \arrow[d, "Fi_-"', shift right] \arrow[d, "Fi_+", shift left] \arrow[d, phantom, "="] \arrow[dr, bend left, "Ff_+"] \arrow[dr, "Ff_-"'] \arrow[dr, phantom, bend left=15, "//"] & \\
F\,Cyl\,A \arrow[r, "Fh"'] & FX
\end{tikzcd}$$
$\boxminus$

Pf of thm. • $f : X \xrightarrow{\sim} Y$ need to find a htpy inverse.

Factor $f : X \overset{\sim}{\underset{i}{\rightarrowtail}} \underset{\underset{\mathcal{M}_{cf}}{\cap}}{Z} \overset{\sim}{\underset{p}{\twoheadrightarrow}} Y$, enough for $i$ & $p$ ... dual, will do for $i$

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, tail, "i"', "\sim"] & X \arrow[d, two heads] \\
Z \arrow[r] \arrow[ur, dashed, "g"] & 1
\end{tikzcd}$$

$g \circ i = 1$
$i \circ g \overset{?}{\sim} 1$

$$\begin{tikzcd}
X \arrow[r, "cst_i"] \arrow[d, tail, "i"', "\sim"] & Path\,Z \arrow[d, two heads] \\
Z \arrow[r, "{(i\circ g, 1)}"'] \arrow[ur, dashed] & Z \times Z
\end{tikzcd}$$

upon composing with $i$
$\underbrace{i \circ g \circ i}_{1} = 1 \circ i$

• the opposite direction is a bit tricky.

$f : X \to Y$ h.e., factor $f : X \overset{\sim}{\underset{i}{\rightarrowtail}} Z \underset{p}{\twoheadrightarrow} Y$
$i$ is a h.e. $\Rightarrow$ $\underline{p \text{ is a h.e.}}$ and we need that it is a w.e.
$s'$ a htpy inverse

$$\begin{tikzcd}
Y \arrow[r, "s'"] \arrow[d, tail, "i_-"', "\sim"] & Z \arrow[d, two heads, "p"] \\
Cyl\,Y \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Y
\end{tikzcd}$$

"homotopy lifting property" $\leadsto$ $\bar{h} : s' \sim s$
such that $s$ is a section, $p \circ s = 1$

$\hookrightarrow h : p \circ s' \sim 1$ ... $s'$ is a "homotopy section"

Still we have $s \circ p \sim 1 \Rightarrow s \circ p$ is a w.e.:

$$\begin{tikzcd}
Z \arrow[d, "i_-"'] \arrow[dr, bend left, "s \circ p"] & \\
Cyl\,Z \arrow[r, "htpy"] & Z \\
Z \arrow[u, "i_+", "\sim"'] \arrow[ur, "1"'] &
\end{tikzcd}$$

General fact: $f_- \sim f_+$ then
$f_-$ w.e. $\Leftrightarrow$ $f_+$ w.e.
$(f_- \sim f_+ \Rightarrow f_- = f_+$ in $\mathcal{W}^{-1}\mathcal{M}$ + saturation$)$

Finally, $p$ is a retract of $s \circ p$, thus a w.e.:

$$\begin{tikzcd}
Z \arrow[r, "1"] \arrow[d, "p"'] & Z \arrow[r, "1"] \arrow[d, "s \circ p"] & Z \arrow[d, "p"] \\
Y \arrow[r, "s"'] & Z \arrow[r, "p"'] & Y
\end{tikzcd}$$

$p \circ s \sim 1 \Rightarrow p \circ s$ w.e.
$s \circ p \sim 1$ $\quad$ $s \circ p$ w.e.
2-out-of-6 $\Downarrow$
$p$ w.e.

"2 out of 6"
$$\begin{tikzcd}
\cdot \arrow[r, "\sim"] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r, "\sim"'] \arrow[ur] & \cdot
\end{tikzcd}$$
$\Rightarrow$ all w.e.

follows from saturation; proved here for the top map identity. $\square$

Construction. Choose, for each $M \in \mathcal{M}$ a cofibrant and fibrant repl.

$$\begin{tikzcd}
M^{cf} & M^c \arrow[l, tail, "\sim"'] \arrow[r, two heads, "\sim"'] & M
\end{tikzcd}$$

since $M^c \xrightarrow{\sim} M^{cf}$; $N^{cf} \xhookleftarrow{\sim} N^f$

Define $Ho(\mathcal{M})\,(M, N) = Ho(\mathcal{M}_{cf})\,(M^{cf}, N^{cf}) = [M^{cf}, N^{cf}] \cong [M^c, N^f]$

$$\begin{tikzcd} M^{cf} & M^{c} \arrow[l, tail, "\sim"'] \arrow[r, two heads, "\sim"'] & M \end{tikzcd}$$

Define $\quad Ho(\mathcal{M})\ (M,N) = Ho(\mathcal{M}_{cf})\left(M^{cf}, N^{cf}\right) = [M^{cf}, N^{cf}] \overset{\downarrow}{\cong} [M^{c}, N^{f}]$

**Theorem.** There is a canonical functor $\mathcal{M} \to Ho(\mathcal{M})$ that displays $Ho(\mathcal{M})$ as the localization $\mathcal{W}^{-1}\mathcal{M}$. In addition, the class $\mathcal{W}$ is saturated.

**Proof.** We will use the "categories" $\mathcal{W}^{-1}\mathcal{M}$ and $\mathcal{W}^{-1}\mathcal{M}_c$.

Assume first functorial cofibrant replacement, call it $Q$

$$\begin{tikzcd} \mathcal{M}_c \arrow[r, hook, shift left, "I"] \arrow[d] & \mathcal{M} \arrow[l, shift left, "Q"] \arrow[d] \\ \mathcal{W}^{-1}\mathcal{M}_c \arrow[r, dashed, shift left, "I'"] & \mathcal{W}^{-1}\mathcal{M} \arrow[l, dashed, shift left, "Q'"] \end{tikzcd}$$

\+ natural transformations / components are we. $\quad QI \xrightarrow{q} 1$ ; $QM \xrightarrow{\sim} M$

$IQ \xrightarrow{q} 1 \quad QA \xrightarrow{\sim} A$ (just a restriction)

\+ natural isomorphisms $\quad Q'I' \xrightarrow{\cong} 1$

$I'Q' \xrightarrow{\cong} 1$

$\Rightarrow$ equivalence of categories

Dually

$$\begin{tikzcd} \mathcal{M}_{cf} \arrow[r, hook] \arrow[d] & \mathcal{M}_c \arrow[l, dashed, bend right, "R"'] \arrow[r, hook] \arrow[d] & \mathcal{M} \arrow[l, dashed, bend right, "Q"'] \arrow[d] \\ \mathcal{W}^{-1}\mathcal{M}_{cf} \arrow[r] & \mathcal{W}^{-1}\mathcal{M}_c \arrow[l, dashed, bend right, "R'"'] \arrow[r] & \mathcal{W}^{-1}\mathcal{M} \arrow[l, dashed, bend right, "Q'"'] \end{tikzcd}$$

$\mathcal{W}^{-1}\mathcal{M}_{cf} \equiv Ho(\mathcal{M}_{cf})$

$\Rightarrow \quad \mathcal{W}^{-1}\mathcal{M}\ (M,N) = \mathcal{W}^{-1}\mathcal{M}_{cf}\ (R'Q'M, R'Q'N) = [M^{cf}, N^{cf}] = Ho(\mathcal{M})$

Saturatedness: $f: M \to N$ gives an iso in $\mathcal{W}^{-1}\mathcal{M}$ iff $RQf$ does

and

$$\begin{tikzcd} M \arrow[d, "f"] & QM \arrow[l, "\sim"', "q"] \arrow[r, "\sim", "r"'] \arrow[d, "Qf"] & RQM \arrow[d, "RQf"] \\ N & QN \arrow[l, "\sim"', "q"] \arrow[r, "\sim", "r"'] & RQN \end{tikzcd}$$

$f$ we $\Leftrightarrow$ $Qf$ w.e. $\Leftrightarrow$ $RQf$ w.e. $\quad \square$

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**Remark.** for non-functorial replacements:

$$\begin{tikzcd} \mathcal{M}_c \arrow[r] \arrow[d] & \mathcal{W}^{-1}\mathcal{M}_c \\ \mathcal{M}_c/\sim_{left} \arrow[ur, dashed] & \end{tikzcd}$$

since left htpic maps are equal in the localization

| this is still a localization at the image of $\mathcal{W}$, i.e. $\mathcal{W}^{-1}(\mathcal{M}_c/\sim_{left})$

$$\begin{tikzcd} M^c \arrow[r, dashed, "\exists"] \arrow[d, two heads, "q"'] & N^c \arrow[d, two heads, "\sim"', "q"] \\ M \arrow[r, mapsto] & N \end{tikzcd}$$

exists and is unique up to left htpy (using lifting axioms)

$\Rightarrow$ get $Q$:

$$\begin{tikzcd} & \mathcal{M}_c \arrow[dl, bend right, "I"'] \arrow[d] \\ \mathcal{M} \arrow[r, "Q"'] & \mathcal{M}_c/\sim_{left} \end{tikzcd}$$

and the induced

$$\begin{tikzcd} W^{-1}M \arrow[r, shift right, "Q'"'] & W^{-1}M_c \arrow[l, shift right, "I'"'] \end{tikzcd}$$

+ natural iso's (exist on loc's only) $\square$

# Quillen functors, derived functors

**Definition.** Let: $F : \mathcal{M} \underset{\perp}{\rightleftarrows} \mathcal{N} : G$ be an adjunction between model categories, $F \dashv G$.

- $F$ is a left Quillen functor if it preserves cofibrations and trivial cofibrations (the "left classes")
- dually $G$ right Quillen
- $F \dashv Q$ a Quillen adjunction if $F$ is l.Q. & $G$ is r.Q.

**Lemma** $F$ l.Q. $\Leftrightarrow$ $G$. r.Q.

**Proof.** $p : X \twoheadrightarrow Y \overset{?}{\Longrightarrow} Gp : GX \twoheadrightarrow GY$ ... $\mathcal{F} = (\mathcal{W} \cap \mathcal{C})^{\square}$ !

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"', "\sim"] & GX \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & GY \end{tikzcd} \equiv \begin{tikzcd} FA \arrow[r] \arrow[d, tail, "i"', "\sim"] & X \arrow[d, two heads] \\ FB \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

F is l.Q.

$\square$

**Remark.** It is enough that $F$ pres. cof. in $\mathcal{M}_c$ and all triv. cof.:

(need $G$ pres. triv. fib.; $p : X \overset{\sim}{\twoheadrightarrow} Y \overset{G \text{ pres fib.}}{\Longleftarrow} Gp : GX \twoheadrightarrow GY$; need w.e.

$$\begin{tikzcd} \mathcal{M}_c \ni A \arrow[r, two heads, "\sim"] \arrow[d, tail] & GX \arrow[d, two heads] \\ B \arrow[r, two heads, "\sim"] \arrow[ur, dashed] & GY \end{tikzcd} \Rightarrow \begin{tikzcd} FA \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "\sim"] \\ FB \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

; apply 2-out-of-6 on the left)

**Construction.**

$$\begin{tikzcd} \mathcal{M} \arrow[r, "FQ"] \arrow[d, "Q"'] & \mathcal{N} \\ \mathcal{M}_c \arrow[r, "F"] & \mathcal{N}_c \arrow[u, hook, "I"'] \end{tikzcd}$$

induces $\rightsquigarrow$ $Ho(\mathcal{M}) \xrightarrow{\mathbb{L}F} Ho(\mathcal{N})$ induced by $FQ$

and dually

$Ho(\mathcal{M}) \xleftarrow{\mathbb{R}G} Ho(\mathcal{N})$ induced by $GR$

preserves w.e. ↑ preserves w.e. by Brown's lemma

**Remark.** $\mathbb{L}F$ is the right Kan extension

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\gamma"] \arrow[d, "F"'] & Ho(\mathcal{M}) \arrow[d, dashed] \\ \mathcal{N} \arrow[r, "\gamma"] & Ho(\mathcal{N}) \end{tikzcd} \quad \mathrm{Ran}_{\gamma}\, \gamma \circ F = \mathbb{L}F$$

**Theorem.** $\mathbb{L}F : Ho(\mathcal{M}) \rightleftarrows Ho(\mathcal{N}) : \mathbb{R}G$ is an adjunction – the derived adjunction.

**Proof.** We need for $M \in \mathcal{M}$, $N \in \mathcal{N}$

$$Ho(\mathcal{N})(\mathbb{L}F M, N) \overset{?}{\cong} Ho(\mathcal{M})(M, \mathbb{R}GN)$$

$$\begin{matrix} \| & & \| \\ [FQM, RN] & & [QM, GRN] \\ \| & & \| \end{matrix}$$

$\mathcal{N}(FQM, RN)/\sim \qquad \overset{\shortparallel}{\mathcal{M}(QM, GRN)/\sim}$

$\longleftarrow$ will happen if the adjunction $F \dashv G$ preserves homotopy, i.e. for $A \in \mathcal{M}_c$, $X \in \mathcal{M}_f$ we need

$$\frac{FA \rightrightarrows X}{A \rightrightarrows GX} \qquad \begin{matrix}\text{htpic} \\ \Updownarrow \\ \text{htpic}\end{matrix}$$

$$\begin{tikzcd}
A + A \arrow[r] \arrow[d, tail] \arrow[dd, bend right=60, "\nabla"'] & GX \\
\mathrm{Cyl}\,A \arrow[ur, dashed, "\text{htpy}"'] \arrow[d, "\sim"] & \\
A &
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
FA + FA \arrow[r] \arrow[d, tail] \arrow[dd, bend right=60, "\nabla"'] & X \\
F\mathrm{Cyl}\,A \arrow[ur, dashed] \arrow[d, "\sim"] & \\
FA &
\end{tikzcd}$$

$\square$

What is the derived unit and the derived counit?

$A \in \mathcal{M}_c \quad \dots \quad A \longrightarrow \mathbb{R}G \circ \mathbb{L}F\, A$ adjoint to $\mathbb{L}FA \xrightarrow{1} \mathbb{L}FA$

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{N})(\mathbb{L}FA, \mathbb{L}FA) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Ho}(\mathcal{M})(A, \mathbb{R}G \circ \mathbb{L}F A) \arrow[d, phantom, "\cong" rotate=90] \\
\mathcal{N}(FQA, RFQA)/\sim \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{M}(QA, GRFQA)/\sim \arrow[d, phantom, "\cong" rotate=90] \\
\mathcal{N}(FA, RFA)/\sim \arrow[r, phantom, "\cong"] & \mathcal{M}(A, GRFA)/\sim
\end{tikzcd}$$

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

the class of
$$\begin{tikzcd}
FA \arrow[r, "1"] \arrow[dr, "\rho"'] & FA \arrow[d, tail, "\sim"] \\
& RFA
\end{tikzcd}$$
its adjoint
$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[rr, bend right, "\eta'"'] & GFA \arrow[r, "Gr_F"] & GRFA
\end{tikzcd}$$

Definition. A Quillen adjunction $F \dashv G$ is said to be a Quillen equivalence if $\mathbb{L}F \dashv \mathbb{R}G$ is an (adjoint) equivalence of categories. This happens iff the derived unit and counit are isomorphisms (on htpy categories), i.e. iff

- $\forall A \in \mathcal{M}_c : \eta' : A \xrightarrow{\eta} GFA \xrightarrow{Gr_F} GRFA$ is a w.e.
- $\forall X \in \mathcal{N}_f : \varepsilon' : FQGX \xrightarrow{Fq_G} FGX \xrightarrow{\varepsilon} X$ is a w.e.

# Small object argument

**Definition.** We say that $A \in \mathcal{M}$ is $\varkappa$-small (a cardinal) if for all $\varkappa$-filtered ordinals $\lambda$ $\mathcal{M}(A,-)$ preserves $\lambda$-indexed colimits

$$\operatorname*{colim}_{\alpha<\lambda} \mathcal{M}(A, M_\alpha) \xrightarrow{\cong} \mathcal{M}(A, \operatorname*{colim}_{\alpha<\lambda} M_\alpha)$$

This means that any subset of cardinality $< \varkappa$ has an upper bound (i.e. a supremum in this case)

E.g. $\aleph_0$-filtered ordinal $\equiv$ limit ordinal

**Example.** A set $A$ is $\varkappa$-small $\Leftrightarrow$ $|A| < \varkappa$.

Any $\varkappa$-presentable object is $\varkappa$-small $\Rightarrow$ in a l.p. cat any object is $\varkappa$-small for some $\varkappa$.

Complication: In Top, the compact Hausdorff spaces are not quite $\aleph_0$-small but the condition holds if the chain $(M_\alpha)_{\alpha<\lambda}$ consists of closed $T_1$-inclusions ($f: X \to Y$ cl. ind., all pts in $Y \setminus f(X)$ closed)

$\llcorner$ all that will be needed

**Construction.** Let $I$ be a set of maps with small domains.

$f: M \to N$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & M \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dashed] & N
\end{tikzcd}$$

probably does not exist but we may adjoin to $M$ a solution:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & M \arrow[d] \arrow[dd, bend left, "f"] \\
B \arrow[r] \arrow[dr, bend right] & M' \arrow[d, dashed] \arrow[ul, phantom, "\ulcorner", very near start] \\
& N
\end{tikzcd}$$

Now take all of them:

$$\square_s = \begin{tikzcd}
A_s \arrow[r] \arrow[d, "i_s"'] & M \arrow[d, "f"] \\
B_s \arrow[r] & N
\end{tikzcd}$$

with $i_s \in I$ indexed by $s \in S$

$$\begin{tikzcd}
\sum\limits_{s\in S} A_s \arrow[r] \arrow[d, "\Sigma i_s"'] & M_0 = M \arrow[d] \arrow[dd, bend left, "f"] \\
\sum\limits_{s\in S} B_s \arrow[r] \arrow[dr, bend right] & M_1 \arrow[d, dashed] \arrow[ul, phantom, "\ulcorner", very near start] \\
& N
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
A \arrow[r] \arrow[dd, "i"'] & M_0 \arrow[d] \\
& M_1 \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & N
\end{tikzcd}$$

since any such square is one of the squares $\square_s$ and then the diagonal is the restriction of the can. map to $B_s$

Proceed inductively $\rightsquigarrow$ factor $M_1 \to N$ as $M_1 \to M_2 \to N$ ... taking $M_\beta = \operatorname*{colim}_{\alpha<\beta} M_\alpha$ for $\alpha$ limit. When do we stop? If the domains of all $i \in I$ are $\varkappa$-small, we stop at any $\varkappa$-filtered limit ordinal $\lambda$.

**Theorem.** The map $M_0 \to M_\lambda$ is a relative $I$-cell complex (is built from $I$ by coproducts, pushouts and transfinite compositions) and the map $M_\lambda \to N$ lies in $I^{\boxslash}$.

**Proof.**

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & M_\lambda \arrow[d] & = \operatorname*{colim}_{\alpha<\lambda} M_\alpha \\
B \arrow[r] & N &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & M_\alpha \arrow[r] & M_{\alpha+1} \arrow[r] & M_\lambda \arrow[d] \\
B \arrow[urr, dashed] \arrow[rrr] & & & N
\end{tikzcd}$$

$\square$