The homotopy category continued

Recall. • Cylinder $\nabla : A+A \overset{i=(i_-,i_+)}{\rightarrowtail} \mathrm{Cyl}\,A \underset{p}{\overset{\sim}{\to}} A$

• Homotopy (left)

$$\begin{tikzcd} A+A \arrow[r, "{[f_-,f_+]}"] \arrow[d, tail, "i"'] & X \\ \mathrm{Cyl}\,A \arrow[ur, "h"'] & \end{tikzcd}$$

• A cofibrant, X fibrant $\Rightarrow$ on $M(A,X)$ left and right htpy agree $\quad M(A,X)/\sim \; = [A,X]$ htpy classes

by using this we require $A \in M_c$, $X \in M_f$

• $Ho(M_{cf}) \overset{\text{def}}{=} M_{cf}/\sim$ ... cofibrant & fibrant objects, htpy classes of maps

$Ho(M_{cf})(A,X) = [A,X]$

quotient category $\quad M_{cf} \to Ho(M_{cf})$ has a universal property

Proposition. A w.e. $f : A \xrightarrow{\sim} B$ induces iso $f^* : [B,X] \to [A,X]$

A w.e. $f : X \xrightarrow{\sim} Y$ induces iso $f_* : [A,X] \to [A,Y]$. $\quad A,B \in M_c$, $X,Y \in M_f$

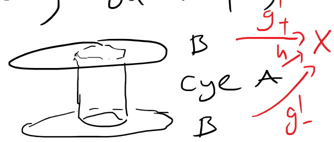
Proof. By Brown's lemma (and duality) $\rightsquigarrow$ enough for $j : A \overset{\sim}{\rightarrowtail} B$

$A \xrightarrow{\sim} \mathrm{Cyl}\,f \xleftarrow{\sim} B$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, tail, "j"', "\sim"] & X \arrow[d] \\ B \arrow[ur, dashed, "g'"] \arrow[r] & 1 \end{tikzcd} \quad \Rightarrow \quad [B,X] \xrightarrow{j^*} [A,X] \text{ surjective (already on maps)},$$

$[g'] \longmapsto [g]$

Construct the "double mapping cylinder" $\mathrm{Cyl}(j,j)$:



$$\begin{tikzcd} A+A \arrow[r, tail, "j+j", "\sim"'] \arrow[d, tail] & B+B \arrow[d, tail] \arrow[rr, bend left, "\nabla"] & & B \\ \mathrm{Cyl}\,A \arrow[r, tail, "\sim"] \arrow[rr, bend right, "p", "\sim"'] & \mathrm{Cyl}(j,j) \arrow[r, tail, "\sim"] \arrow[ul, phantom, "\ulcorner"] & \mathrm{Cyl}\,B \arrow[r, "\sim"] & B \\ & & A \arrow[ur, bend right, "j", "\sim"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Cyl}(j,j) \arrow[r, "{[g'_-,g'_+,h]}"] \arrow[d, tail, "\sim"'] & X \\ \mathrm{Cyl}\,B \arrow[ur, dashed, "k"'] & \end{tikzcd}$$

$\mathrm{Cyl}(j,j)$

$\square$

Theorem ("Whitehead"). $A, X \in M_{cf}$, $f : A \to X$; then

f is a w.e. $\iff$ f is a h.e.

$\llcorner$ homotopy equivalence $=$ iso in $Ho(M_{cf})$

Corollary. The canonical (projection) functor

$M_{cf} \to Ho(M_{cf})$

presents $Ho(M_{cf})$ as the localization $W^{-1}M_{cf}$.

In addition, the class W is saturated in M_{cf} in that it consists of all maps that get inverted in $W^{-1}M_{cf}$.

Pf of Cor.

$$\begin{tikzcd} & Ho(M_{cf}) = M_{cf}/\sim \arrow[d, dashed] \\ M_{cf} \arrow[ur] \arrow[r, "F"'] & K \end{tikzcd} \quad \leftarrow \text{ has univ. prop.}$$

then the functor exists

We need to show that $F(W) \subseteq \mathrm{Iso}(K)$ $\Longrightarrow$ $f_- \sim f_+ \Rightarrow Ff_- = Ff_+$.

$A \xrightarrow{1} A$

We need to show that $F(\mathcal{W}) \subseteq \mathrm{Iso}(\mathcal{K}) \Rightarrow f_- \sim f_+ \Rightarrow Ff_- = Ff_+$.

$$\begin{tikzcd}
FA \arrow[d, "Fi_-"'] \arrow[dr, "1"] & \\
F\,\mathrm{Cyl}\,A \arrow[r, "Fp", "\cong"'] & FA \\
FA \arrow[u, "Fi_+"] \arrow[ur, "1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "i_-"'] \arrow[r, "1"] & A \\
\mathrm{Cyl}\,A \arrow[ur, "p", "\sim"'] & \\
A \arrow[u, "i_+"] \arrow[uur, "1"'] &
\end{tikzcd}$$

Fi_-, Fi_+ right inverses of an iso Fp

$\Rightarrow Fi_- = Fi_+ \Rightarrow$

$$\begin{tikzcd}
FA \arrow[d, "Fi_-"', shift right] \arrow[d, "Fi_+", shift left] \arrow[dr, bend left, "Ff_+"] \arrow[dr, "Ff_-"'] & \\
F\,\mathrm{Cyl}\,A \arrow[r, "Fh"'] & FX
\end{tikzcd}$$

$\square$

Pf of thm. • $f : X \xrightarrow{\sim} Y$ need to find a htpy inverse.

Factor $f : X \overset{\sim}{\underset{i}{\rightarrowtail}} Z \overset{\sim}{\underset{p}{\twoheadrightarrow}} Y$, $Z \in \mathcal{M}_{cf}$. ~~)~~ enough for i & p ... dual, will do for i

Prop. $f^* : [Y, X] \xrightarrow{\cong} [X, X]$, $g \mapsto 1$ (htpy inv.)

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "i", "\sim"'] & X \arrow[d] \\
Z \arrow[r] \arrow[ur, dashed, "g"] & 1
\end{tikzcd}
\qquad
\begin{array}{l} g \circ i = 1 \\ i \circ g \overset{?}{\simeq} 1 \end{array}
\qquad
\begin{tikzcd}
X \arrow[r, "\mathrm{cst}_i"] \arrow[d, "i", "\sim"'] & \mathrm{Path}\,Z \arrow[d] \\
Z \arrow[r, "{(iog, 1)}"'] \arrow[ur, dashed] & Z \times Z
\end{tikzcd}$$

upon composing with i: $\underbrace{i o g o i}_{1} = 1 o i$

• the opposite direction is a bit tricky.

$f : X \to Y$ h.e., factor $f : X \overset{\sim}{\underset{i}{\rightarrowtail}} Z \underset{p}{\twoheadrightarrow} Y$

i is a h.e. $\Rightarrow$ $\underline{p \text{ is a h.e.}}$ and we need that it is a w.e.

s' a htpy inverse

$$\begin{tikzcd}
Y \arrow[r, "s'"] \arrow[d, "i_-", "\sim"'] & Z \arrow[d, "p"] \\
\mathrm{Cyl}\,Y \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Y
\end{tikzcd}$$

"homotopy lifting property" $\rightsquigarrow$ $\bar{h} : s' \sim s$ such that s is a section, $p o s = 1$

$\llcorner$ $h : p o s' \sim 1$ ... s' is a "homotopy section"

Still we have $s o p \sim 1 \Rightarrow s \circ p$ is a w.e.:

$s o p \sim s' o p$

$$\begin{tikzcd}
Z \arrow[d, "i_-", "\sim"'] \arrow[dr, "s o p"] & \\
\mathrm{Cyl}\,Z \arrow[r, "\text{htpy}", "\sim"'] & Z \\
Z \arrow[u, "i_+", "\sim"'] \arrow[ur, "1"'] &
\end{tikzcd}$$

General fact: $f_- \sim f_+$ then f_- w.e. $\Leftrightarrow$ f_+ w.e.

($f_- \sim f_+ \Rightarrow f_- = f_+$ in $\mathcal{W}^{-1}\mathcal{M}$ + saturation)

Finally, p is a retract of $s o p$, thus a w.e.:

$$\begin{tikzcd}
Z \arrow[r, "1"] \arrow[d, "p"] & Z \arrow[r, "1"] \arrow[d, "s o p"] & Z \arrow[d, "p"] \\
Y \arrow[r, "s"'] & Z \arrow[r, "p"'] & Y
\end{tikzcd}$$

$p o s o p = p$

$p o s = 1$

"2 out of 6"

$p o s \sim 1 \Rightarrow p o s$ w.e.

$s o p \sim 1 \Rightarrow s o p$ w.e.

2-out-of-6 $\Downarrow$

p w.e.

$$\begin{tikzcd}
\cdot \arrow[r, "\sim"] \arrow[dr, "p"] \arrow[d, "s"'] & \cdot \arrow[d, "s"] \\
\cdot \arrow[r, "\sim"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "\sim"] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r, "\sim"'] \arrow[ur] & \cdot
\end{tikzcd}
\Rightarrow \text{all w.e.}$$

follows from saturation; proved here for the top map identity. $\square$

Construction. Choose, for each $M \in \mathcal{M}$ a cofibrant and fibrant repl.

$M^{cf} \xleftarrow{\sim} M^c \xrightarrow{\sim} M$

since $M^c \xrightarrow{\sim} M^{cf}$; $N^{cf} \xleftarrow{\sim} N^f$

Define $\mathrm{Ho}(\mathcal{M})(M, N) = \mathrm{Ho}(\mathcal{M}_{cf})(M^{cf}, N^{cf}) = [M^{cf}, N^{cf}] \overset{\downarrow}{\cong} [M^c, N^f]$

$$M^{cf} \xleftarrow{\sim} M^c \xrightarrow{\sim} M$$

Define $\quad Ho(\mathcal{M})\ (M,N) = Ho(\mathcal{M}_{cf})\left(M^{cf}, N^{cf}\right) = [M^{cf}, N^{cf}] \cong [M^c, N^f]$

Theorem. There is a canonical functor $\mathcal{M} \to Ho(\mathcal{M})$ that displays $Ho(\mathcal{M})$ as the localization $\mathcal{W}^{-1}\mathcal{M}$.
In addition, the class $\mathcal{W}$ is saturated. $\Rightarrow$ 2-out-of-6

Proof. We will use the "categories" $\mathcal{W}^{-1}\mathcal{M}$ ← gen. $\mathcal{M}$ + inverzemi $\mathcal{W}$ and $\mathcal{W}^{-1}\mathcal{M}_c$

Assume first functorial cofibrant replacement, call it Q

$$\begin{tikzcd} QX \arrow[r, "Qf"] \arrow[d, "q"', "\sim"] & QY \arrow[d, "q", "\sim"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{M}_c \arrow[r, shift left, "I"] \arrow[d] & \mathcal{M} \arrow[l, shift left, "Q"] \arrow[d] \\ \mathcal{W}^{-1}\mathcal{M}_c \arrow[r, dashed, shift left, "I'"] & \mathcal{W}^{-1}\mathcal{M} \arrow[l, dashed, shift left, "Q'"] \end{tikzcd}$$

+ natural transformations $QI \xrightarrow{q} 1$, $IQ \xrightarrow{q} 1$ (just a restriction) ; $QM \xrightarrow{\sim} M$, $QA \xrightarrow{\sim} A$
components are w.e.

+ natural isomorphisms $Q'I' \xrightarrow{\cong} 1$, $I'Q' \xrightarrow{\cong} 1$

$\Rightarrow$ equivalence of categories

Dually

$$\begin{tikzcd} \mathcal{M}_{cf} \arrow[r, shift right] \arrow[d] & \mathcal{M}_c \arrow[l, dashed, shift right, "R"'] \arrow[r, shift right] \arrow[d] & \mathcal{M} \arrow[l, dashed, shift right, "Q"'] \arrow[d] \\ \mathcal{W}^{-1}\mathcal{M}_{cf} \arrow[r, shift right] & \mathcal{W}^{-1}\mathcal{M}_c \arrow[l, dashed, shift right, "R'"'] \arrow[r, shift right] & \mathcal{W}^{-1}\mathcal{M} \arrow[l, dashed, shift right, "Q'"'] \end{tikzcd}$$

$\mathcal{W}^{-1}\mathcal{M}_{cf} \equiv Ho(\mathcal{M}_{cf})$

$\Rightarrow \quad \mathcal{W}^{-1}\mathcal{M}\,(M,N) = \mathcal{W}^{-1}\mathcal{M}_{cf}\,(R'Q'M, R'Q'N) = [M^{cf}, N^{cf}] = Ho(\mathcal{M})(M,N)$

Saturatedness: $f : M \to N$ gives an iso in $\mathcal{W}^{-1}\mathcal{M}$ iff RQf does

and
$$\begin{tikzcd} M \arrow[d, "f"] & QM \arrow[l, "\sim"', "q"] \arrow[r, "\sim", "r"'] \arrow[d, "Qf"] & RQM \arrow[d, "RQf"] \\ N & QN \arrow[l, "\sim"', "q"] \arrow[r, "\sim", "r"'] & RQN \end{tikzcd}$$
f we $\Leftrightarrow$ Qf w.e. $\Leftrightarrow$ RQf w.e. $\square$

Remark. for non-functorial replacements:

$$\begin{tikzcd} \mathcal{M}_c \arrow[r] \arrow[d] & \mathcal{W}^{-1}\mathcal{M}_c \\ \mathcal{M}_c/\sim_{left} \arrow[ur, dashed] & \end{tikzcd}$$

← since left htpic maps are equal in the localization | this is still a localization at the image of $\mathcal{W}$, i.e. $\mathcal{W}^{-1}(\mathcal{M}_c/\sim_{left})$

$$\begin{tikzcd} M^c \arrow[r, dashed, "\exists"] \arrow[d, "q"'] & N^c \arrow[d, "q", "\sim"'] \\ M \arrow[r, mapsto] & N \end{tikzcd}$$
exists and is unique up to left htpy (using lifting axioms)

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & N^c \arrow[d, "\sim"] \\ M^c \arrow[r, mapsto] \arrow[ur, dashed, "f^c"] & M \arrow[r, mapsto, "f"] & N \end{tikzcd}$$

$\Rightarrow$ get Q :
$$\begin{tikzcd} & \mathcal{M}_c \arrow[ld, "I"'] \arrow[d] \\ \mathcal{M} \arrow[r, "Q"] & \mathcal{M}_c/\sim_{left} \end{tikzcd}$$

$M^c \to M \xrightarrow{f} N$, t, $\sim$

$$Q \simeq M_c/\sim_{left}$$

and the induced

$$W^{-1}M \underset{Q'}{\overset{I'}{\leftrightarrows}} W^{-1}M_c$$

+ natural iso's (exist on loc's only) $\square$

Quillen functors, derived functors

Definition. Let: $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be an adjunction between model categories, $F \dashv G$.

- F is a left Quillen functor if it preserves cofibrations and trivial cofibrations (the "left classes")
- dually G right Quillen
- $F \dashv G$ a Quillen adjunction if F is l.Q. & G is r.Q.

Lemma F l.Q. $\Leftrightarrow$ G r.Q.

Proof. $p : X \twoheadrightarrow Y \overset{?}{\Rightarrow} Gp : GX \twoheadrightarrow GY$ ... $\mathcal{F} = (\mathcal{W} \cap \mathcal{C})^{\square}$?

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"', "\sim"] & GX \arrow[d, "Gp"] \\ B \arrow[r] \arrow[ur, dashed] & GY \end{tikzcd} \quad \equiv \quad \begin{tikzcd} FA \arrow[r] \arrow[d, "\sim"'] & X \arrow[d, "p"] \\ FB \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

F is l.Q.

Podobně: F zach. lof $\Leftrightarrow$ G zach. triv. fib. □

Remark. It is enough that F pres. cof. in $\mathcal{M}_c$ and all triv. cof.:

(need G pres. triv. fib.; $p : X \overset{\sim}{\twoheadrightarrow} Y \overset{G \text{ pres fib.}}{\Leftarrow} Gp : GX \twoheadrightarrow GY$; need w.e.

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[dr, tail] & A \arrow[r, "\sim"] \arrow[d, tail] & GX \arrow[d, "Gp"] \\ & B \arrow[r, "\sim"] \arrow[ur, dashed] & GY \end{tikzcd} \Rightarrow \begin{tikzcd} FA \arrow[r] \arrow[d, tail] & X \arrow[d, "\sim"] \\ FB \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$\mathcal{M}_c \ni$; lof. vzh. GX

; apply 2-out-of-6 on the left)

Prop. 8.5.4.

Construction.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "FQ"] \arrow[d, "Q"'] & \mathcal{N} \\ \mathcal{M}_c \arrow[r, "F"] & \mathcal{N}_c \arrow[u, "I"'] \end{tikzcd}$$

induces $\rightsquigarrow$ $Ho(\mathcal{M}) \xrightarrow{\mathbb{L}F} Ho(\mathcal{N})$ induced by FQ

and dually

$Ho(\mathcal{M}) \xleftarrow{\mathbb{R}G} Ho(\mathcal{N})$ induced by GR

preserves w.e. ; preserves w.e. by Brown's lemma $\leftarrow$ F zach. w.e. mezi lof.

Remark. $\mathbb{L}F$ is the right Kan extension

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\gamma"] \arrow[d, "F"'] & Ho(\mathcal{M}) \arrow[d, dashed] \\ \mathcal{N} \arrow[r, "\gamma"] & Ho(\mathcal{N}) \end{tikzcd} \quad \mathrm{Ran}_\gamma\, \gamma \circ F = \mathbb{L}F$$

Theorem. $\mathbb{L}F : Ho(\mathcal{M}) \rightleftarrows Ho(\mathcal{N}) : \mathbb{R}G$ is an adjunction — the derived adjunction.

Proof. We need for $M \in \mathcal{M}$, $N \in \mathcal{N}$

$$Ho(\mathcal{N})(\mathbb{L}FM, N) \overset{?}{\cong} Ho(\mathcal{M})(M, \mathbb{R}GN)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$[FQM, RN] \qquad\qquad [QM, GRN]$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\mathcal{N}(FQM, RN)/\sim \qquad\qquad \mathcal{M}(QM, GRN)/\sim$$

Q není potřeba, protože FQM je lof.

will happen if the adjunction $F \dashv G$ preserves homotopy, i.e. for $A \in \mathcal{M}_c$, $X \in \mathcal{M}_f$ we need

F zach. kolimity – je to l.adj.: $F(A+A)$

$FA \longrightarrow X$ htopic $\quad A + A \longrightarrow GX \quad FA + FA \longrightarrow X$

preserves ^hof^ homotopy, i.e. for $A \in M_c$, $X \in M_f$ we need $F(A+A)$ $=$

$$\frac{FA \rightrightarrows X \quad \text{htpic}}{A \rightrightarrows GX \quad \text{htpic}} \qquad
\begin{tikzcd}
A+A \arrow[r] \arrow[d, tail] \arrow[dd, bend right, "\nabla"'] & GX \\
\mathrm{Cyl}\,A \arrow[d, "\sim"] \arrow[ur, dashed, "\text{htpy}"'] & \\
A &
\end{tikzcd}
\equiv
\begin{tikzcd}
FA+FA \arrow[r] \arrow[d, tail] \arrow[dd, bend right, "\nabla"'] & X \\
F\mathrm{Cyl}\,A \arrow[d, "\sim"] \arrow[ur, dashed] & \\
FA &
\end{tikzcd}$$

lepší $\quad A \xrightarrow{i_-} \mathrm{Cyl}\,A \xleftarrow{i_+} A$, $\quad \mathrm{Cyl}\,A \xrightarrow{p} A$, $\quad 1, 1$

w.e. mezi hof. obj.

Brownovo lemma

je to htpie

je to cylindr $\square$

What is the derived unit and the derived counit ?

$A \in M_c$ ... $A \longrightarrow \mathbb{R}G \circ \mathbb{L}F\, A$ adjoint to $\mathbb{L}FA \xrightarrow{1} \mathbb{L}FA$

$$1 \in Ho(N)(\mathbb{L}FA, \mathbb{L}FA) \equiv Ho(M)(A, \mathbb{R}G \circ \mathbb{L}F\,A)$$

$$\begin{array}{ccc}
Ho(N)(\mathbb{L}FA, \mathbb{L}FA) & \equiv & Ho(M)(A, \mathbb{R}G \circ \mathbb{L}F\,A) \\
\wr\| & & \wr\| \\
N(FQA, RFQA)/\sim & \cong & M(QA, GRFQA)/\sim \\
\wr\| & & \wr\| \\
r \in N(FA, RFA)/\sim & \cong & M(A, GRFA)/\sim \\
\Downarrow & & \Downarrow
\end{array}$$

the class of
$$\begin{tikzcd}
FA \arrow[r, "1"] \arrow[d, "\sim"'] & FA \arrow[d, "\sim"] \\
rF & RFA
\end{tikzcd}$$
its adjoint
$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[rr, bend right, "\eta'"'] & GFA \arrow[r, "GrF"] & GRFA
\end{tikzcd}$$

Definition. A Quillen adjunction $F \dashv G$ is said to be a **Quillen equivalence** if $\mathbb{L}F \dashv \mathbb{R}G$ is an (adjoint) equivalence of categories. This happens iff the derived unit and counit are isomorphisms (on htpy categories), i.e. iff

- $\forall A \in M_c$: η': $A \xrightarrow{\eta} GFA \xrightarrow{GrF} GRFA$ is a w.e.
- $\forall X \in N_f$: ε': $FQGX \xrightarrow{FqG} FGX \xrightarrow{\varepsilon} X$ is a w.e.

Small object argument

Definition. We say that $A \in \mathcal{M}$ is $\varkappa$-small (a cardinal) if, for all $\varkappa$-filtered limit ordinals λ, $\mathcal{M}(A,-)$ preserves λ-indexed colimits

$$\operatorname*{colim}_{\alpha<\lambda} \mathcal{M}(A, M_\alpha) \xrightarrow{\cong} \mathcal{M}(A, \operatorname*{colim}_{\alpha<\lambda} M_\alpha)$$

this means that any subset of cardinality $< \varkappa$ has an upper bound (i.e. a supremum in this case)

E.g. $\aleph_0$-filtered ordinal $\equiv$ limit ordinal

Example. A set A is $\varkappa$-small $\iff$ $|A| < \varkappa$.

Any $\varkappa$-presentable object is $\varkappa$-small $\Rightarrow$ in a l.p. cat any object is $\varkappa$-small for some $\varkappa$.

Complication: In Top, the compact Hausdorff spaces are not quite $\aleph_0$-small but the condition holds if the chain $(M_\alpha)_{\alpha<\lambda}$ consists of closed T_1-inclusions ($f: X \to Y$ cl. ind., all pts in $Y \setminus f(X)$ closed)

$\llcorner$ all that will be needed

Construction. Let $\mathcal{I}$ be a set of maps with small domains.

$f : M \to N$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & M \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

probably does not exist but we may adjoin to M a solution:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & M \arrow[d] \arrow[dd, bend left, "f"] \\ B \arrow[r] \arrow[dr] & M' \arrow[d, dashed] \\ & N \end{tikzcd}$$

Now take all of them:

$$\square_s = \begin{tikzcd} A_s \arrow[r] \arrow[d, "i_s"'] & M \arrow[d, "f"] \\ B_s \arrow[r] & N \end{tikzcd}$$ with $i_s \in \mathcal{I}$ indexed by $s \in S$

$$\begin{tikzcd} \sum_{s\in S} A_s \arrow[r] \arrow[d, "\sum i_s"'] & M_0 = M \arrow[d] \arrow[dd, bend left, "f"] \\ \sum_{s\in S} B_s \arrow[r] \arrow[dr] & M_1 \arrow[d, dashed] \\ & N \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & M_0 \arrow[d] \\ & M_1 \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

since any such square is one of the squares $\square_s$ and then the diagonal is the restriction of the can. map to B_s

Proceed inductively $\leadsto$ factor $M_1 \to N$ as $M_1 \to M_2 \to N$ ... taking $M_\beta = \operatorname*{colim}_{\alpha<\beta} M_\alpha$ for β limit. When do we stop? If the domains of all $i \in \mathcal{I}$ are $\varkappa$-small, we stop at any $\varkappa$-filtered limit ordinal λ.

Theorem. The map $M_0 \to M_\lambda$ is a relative I-cell complex (is built from I by coproducts, pushouts and transfinite compositions) and the map $M_\lambda \to N$ lies in $I^{\boxslash}$.

Proof.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & M_\lambda = \operatorname*{colim}_{\alpha<\lambda} M_\alpha \arrow[d] \\ B \arrow[r] & N \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & M_\alpha \arrow[r] & M_{\alpha+1} \arrow[r] & M_\lambda \arrow[d] \\ B \arrow[urr, dashed] \arrow[rrr] & & & N \end{tikzcd}$$

$\square$