# Quillen functors – reminder

A Quillen adjunction is an adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$, $F \dashv G$ between model categories s.t.

F pres. cofibrations $\Leftrightarrow$ G pres. triv. fibrations

F pres. triv. cofibrations $\Leftrightarrow$ G pres. fibrations

(enough: fibrations between fibrant objects)

Definition. A Quillen adjunction $F \dashv G$ is said to be a Quillen equivalence if $\mathbb{L}F \dashv \mathbb{R}G$ is an (adjoint) equivalence of categories. This happens iff the derived unit and counit are isomorphisms (on htpy categories), i.e. iff

- $\forall A \in \mathcal{M}_c$ : $\eta' : A \xrightarrow{\eta} GFA \xrightarrow{GrF} GRFA$ is a w.e. (derived unit)
- $\forall X \in \mathcal{N}_f$ : $\varepsilon' : FQGX \xrightarrow{FqG} FGX \xrightarrow{\varepsilon} X$ is a w.e. (derived counit)

Later. Quillen bifunctors, monoidal model categories, enriched model categories

# Small object argument

**Definition.** We say that $A \in \mathcal{M}$ is $\varkappa$-small (a cardinal) if, for all $\varkappa$-filtered limit ordinals $\lambda$, $\mathcal{M}(A,-)$ preserves $\lambda$-indexed colimits

$$\operatorname{colim}_{\alpha<\lambda} \mathcal{M}(A, M_\alpha) \xrightarrow{\cong} \mathcal{M}(A, \operatorname{colim}_{\alpha<\lambda} M_\alpha)$$

this means that any subset of cardinality $< \varkappa$ has an upper bound (i.e. a supremum in this case)

E.g. $\aleph_0$-filtered ordinal $\equiv$ limit ordinal

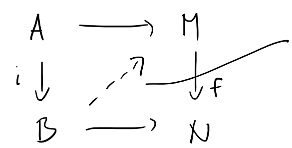
**Example.** $A$ set $A$ is $\varkappa$-small $\Leftrightarrow$ $|A| < \varkappa$.

Any $\varkappa$-presentable object is $\varkappa$-small $\Rightarrow$ in a l.p. cat any object is $\varkappa$-small for some $\varkappa$.

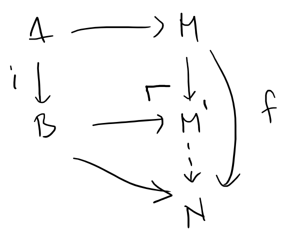
Complication: In Top, the compact Hausdorff spaces are not quite $\aleph_0$-small but the condition holds if the chain $(M_\alpha)_{\alpha<\lambda}$ consists of closed $T_1$-inclusions ($f: X \to Y$ cl. ind., all pts in $Y \setminus f(X)$ closed)
↳ all that will be needed

**Construction.** Let $I$ be a set of maps with small domains.

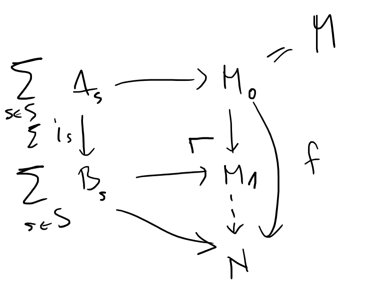
$f : M \to N$



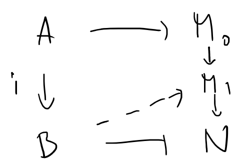

probably does not exist but we may adjoin to $M$ a solution:




Now take all of them: $\square_s = \begin{array}{ccc} A_s & \to & M \\ i_s\downarrow & & \downarrow f \\ B_s & \to & N \end{array}$ with $i \in I$ indexed by $s \in S$




$\Rightarrow$

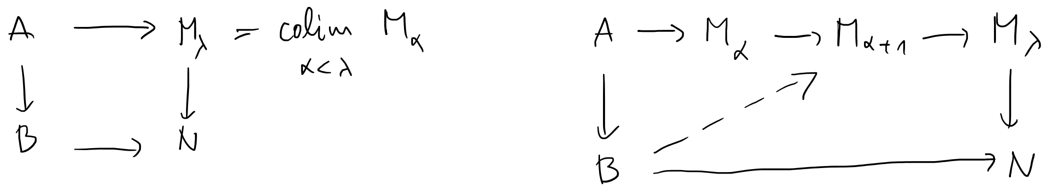



since any such square is one of the squares $\square_s$ and then the diagonal is the restriction of the can. map to $B_s$

Proceed inductively $\rightsquigarrow$ factor $M_1 \to N$ as $M_1 \to M_2 \to N$ ... taking $M_\beta = \operatorname{colim}_{\alpha<\beta} M_\alpha$ for $\alpha$ limit. When do we stop? If the domains of all $i \in I$ are $\varkappa$-small, we stop at any $\varkappa$-filtered limit ordinal $\lambda$.

**Theorem.** The map $M_0 \to M_\lambda$ is a relative $\mathcal{I}$-cell complex (is built from $\mathcal{I}$ by coproducts, pushouts and transfinite compositions) and the map $M_\lambda \to N$ lies in $\mathcal{I}^\square$.

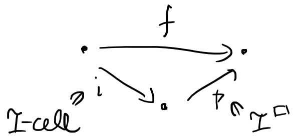
**Proof.**



$\square$

**Corollary.** The set $\mathcal{I}$ generates a weak factorization system $(\mathcal{L}, \mathcal{R}) = ({}^\square(\mathcal{I}^\square), \mathcal{I}^\square)$, i.e. $\mathcal{L} \square \mathcal{R}$, $\mathcal{M} = \mathcal{R} \circ \mathcal{L}$.
Moreover, ${}^\square(\mathcal{I}^\square)$ consists precisely of retracts of relative $\mathcal{I}$-cell cxs.

**Proof.** $\mathcal{L} \square \mathcal{R}$ by definition, $\mathcal{M} = \mathcal{R} \circ \mathcal{L}$ by the theorem. Let $f \in {}^\square(\mathcal{I}^\square)$ and factor it as in the theorem:



retract argument $\Longrightarrow$ $f \square p$; $f$ is a retract of $i$. $\square$

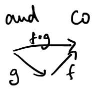
This gives a way of constructing examples of model categories

- $\mathcal{M}$ complete & cocomplete
- $\mathcal{W}$ closed under retracts & 2-out-of-3
- $\mathcal{F}$ a class of fibrations
- $\mathcal{W} \cap \mathcal{F} = \mathcal{I}^\square$ $\longrightarrow$ this defines $\mathcal{C} = {}^\square(\mathcal{W} \cap \mathcal{F}) = {}^\square(\mathcal{I}^\square)$ = retracts of relative $\mathcal{I}$-cell cxs
- $\mathcal{F} = \mathcal{J}^\square$

both $\mathcal{I}$ & $\mathcal{J}$ sets of maps with small domains

What needs to be proved? We get w.f.s.'s $({}^\square(\mathcal{I}^\square), \mathcal{I}^\square) = (\mathcal{C}, \mathcal{W} \cap \mathcal{F})$
$({}^\square(\mathcal{J}^\square), \mathcal{J}^\square) \overset{?}{=} (\underline{\underline{\mathcal{W} \cap \mathcal{C}}}, \mathcal{F})$

**Recall:**
- MC1 (finite) limits and colimits exist in $\mathcal{M}$
- MC2 2-out-of-3: 2 of these in $\mathcal{W}$ $\Rightarrow$ so is 3rd for $\mathcal{W}$



- MC3 All $\mathcal{W}, \mathcal{C}, \mathcal{F}$ are closed under retracts
- MC4 $\mathcal{C} \square (\mathcal{W} \cap \mathcal{F})$, $(\mathcal{W} \cap \mathcal{C}) \square \mathcal{F}$
- MC5 $\mathcal{M} = (\mathcal{W} \cap \mathcal{F}) \circ \mathcal{C}$, $\mathcal{M} = \mathcal{F} \circ (\mathcal{W} \cap \mathcal{C})$

$(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ & $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$ form the so-called weak factorization systems

**Theorem.** Assume as above that $\mathcal{M}$ is bicomplete, $\mathcal{W}$ closed under retracts and 2-out-of-3 and $\mathcal{W} \cap \mathcal{F} = \mathcal{I}^\square$, $\mathcal{F} = \mathcal{J}^\square$ for some sets $\mathcal{I}, \mathcal{J}$ of maps with small domains.
Then $\mathcal{M}$ is a model category iff ${}^\square(\mathcal{J}^\square) \subseteq \mathcal{W}$.
We say that $\mathcal{M}$ is *cofibrantly generated*. ($\mathcal{J}$-cell sufficient)

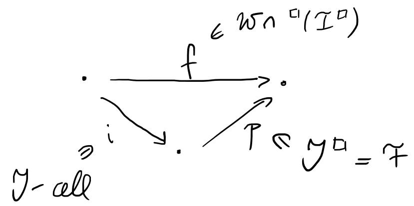
**Proof.** We have $\mathcal{F} \supseteq \mathcal{W} \cap \mathcal{F} \Rightarrow {}^\square(\mathcal{J}^\square) \subseteq {}^\square(\mathcal{I}^\square)$, ...

so that in fact ${}^{\square}(\mathcal{J}^{\square}) \subseteq \mathcal{W} \cap {}^{\square}(\mathcal{I}^{\square})$. We have seen that

$${}^{\square}(\mathcal{J}^{\square}) = \mathcal{W} \cap {}^{\square}(\mathcal{I}^{\square})$$

is sufficient, so we proceed to show "$\supseteq$".

Thus, let $f \in \mathcal{W} \cap {}^{\square}(\mathcal{I}^{\square})$ and factor it using SOA w.r.t. $\mathcal{J}$:




$i \in \mathcal{J}\text{-cell} \subseteq \mathcal{W} \Rightarrow p \in \mathcal{W}$ by 2-out-of-3

$\Rightarrow p \in \mathcal{W} \cap \mathcal{F}$

$\Rightarrow f \,\square\, p$ $\xRightarrow[\text{argument}]{\text{retract}}$ $f$ is a retract of $i$

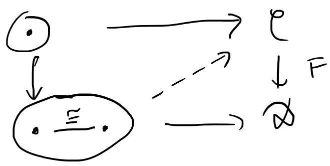
$\Rightarrow f \in {}^{\square}(\mathcal{J}^{\square})$ $\square$

# Examples

One of the simplest examples is $\mathcal{M} = \mathcal{C}at$

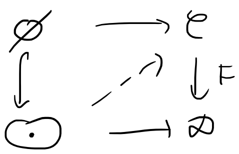
$\mathcal{W}$ = equivalences of categories

$\mathcal{F}$ = isofibrations = $F : \mathcal{C} \to \mathcal{D}$ s.t. given $c \in \mathcal{C}$ and an iso $Fc \cong d$ a lift $c \cong c'$ exists



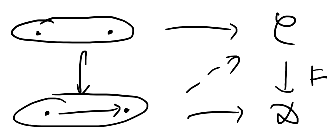

$\rightsquigarrow \mathcal{J} = \{\{0\} \hookrightarrow \{0 \cong 1\}\}$

$\mathcal{W} \cap \mathcal{F}$ = surjective equivalences of categories
↳ ess. surj. on objects; if iso-fib ⇒ surj.

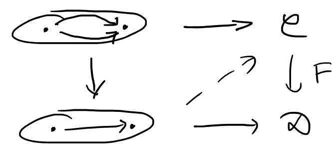



← surjectivity on objects →

surjectivity on maps →




injectivity on maps →




$\rightsquigarrow \mathcal{I} = \{\emptyset \hookrightarrow \{0\}, \{0 \;\; 1\} \hookrightarrow \{0 \to 1\}, \{0 \rightrightarrows 1\} \to \{0 \to 1\}\}$

Clearly: $\mathcal{J}$-cell = injective equivalences $= {}^{\square}(\mathcal{J}^{\square})$
$\mathcal{I}$-cell = functors injective on objects $= {}^{\square}(\mathcal{I})^{\square}$ $\Big\} \Rightarrow {}^{\square}(\mathcal{J}^{\square}) = \mathcal{W} \cap {}^{\square}(\mathcal{I}^{\square})$

---

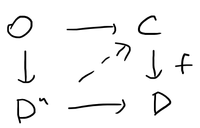
$\mathcal{M} = Ch$, say non-negatively graded chain complexes of (right) $R$-modules

$\mathcal{W}$ = quasi-isomorphisms

$\mathcal{F}$ = maps that are surjective in positive dimensions

Define for $n > 0$: $D^n = ( \cdots \to 0 \to \overset{n}{R} \xrightarrow{1} \overset{n-1}{R} \to 0 \to \cdots)$

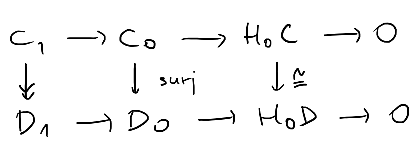
$Ch(D^n, C) \cong C_n$ so that




$\iff f \in \mathcal{F}$

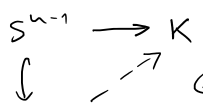
$\mathcal{W} \cap \mathcal{F}$ = surjective quasi-iso's ; $f : C \to D$ induces

= kernel acyclic




Define for $n > 0$: $S^{n-1} = ( \cdots \to 0 \to \overset{n}{0} \to \overset{n-1}{R} \to 0 \to \cdots ) \subseteq D^n$

$Ch(S^{n-1}, C) \cong Z_{n-1} C$ $(n-1)$-cycles




$\iff H_{n-1} K = 0$

$S^{n-2} \quad S^{n-1} \quad S^{n-1} \longrightarrow D^n$ Remark. $n = 0$

$\mathrm{Ch}(S^{n-1}, C) \cong Z_{n-1} C$ $(n-1)$-cycles

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & K \\ \downarrow & \nearrow & \\ D^n & & \end{array} \iff H_{n-1}K = 0$$

$$\begin{array}{ccccccc} S^{n-2} & \to & D^{n-1} & S^{n-1} & \longrightarrow & D^n \\ 0 \downarrow & \text{po} & \downarrow 1 & & \ulcorner & \downarrow 1 \\ 0 & \longrightarrow & S^{n-1} & & \longrightarrow & D^n \end{array}$$

Remark. $n=0$

$S^{-1} \to D^0$ $\quad Z_{-1}C = 0$

$= 0 \to S^0$ $\quad C_0 = Z_0 C$

$\Rightarrow \mathcal{I}^\square \subseteq \mathcal{J}^\square =$ surj's in pos. dim's

$\rightsquigarrow \mathcal{I}^\square \subseteq \mathcal{W} \cap \mathcal{J}^\square$ since

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{z} & C \\ \downarrow & \nearrow & \downarrow f \\ D^n & \xrightarrow[0]{} & D \end{array} \quad \equiv \quad \begin{array}{ccc} S^{n-1} & \xrightarrow{z} & \ker f \\ \downarrow & \nearrow & \\ D^n & & \end{array} \iff H_{n-1}\ker f = 0$$

For the opposite direction: let $f \in \mathcal{W} \cap \mathcal{J}^\square$, i.e. $f$ is a surj. quasi-iso

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{z} & C \\ \downarrow & \nearrow & \downarrow f \\ D^n & \xrightarrow[d]{} & D \end{array} \; - \; \begin{array}{ccc} S^{n-1} & \xrightarrow{\partial c} & C \\ \downarrow & & \downarrow f \\ D^n & \xrightarrow[d]{} & D \end{array} \; = \; \begin{array}{ccc} S^{n-1} & \xrightarrow{z-\partial c} & C \\ \downarrow & \nearrow & \downarrow f \\ D^n & \xrightarrow[0]{} & D \end{array}$$

has a solution since $\ker f$ is acyclic

↑ by surjectivity, $c$ exists

Summary. $\mathcal{I} = \{ S^{n-1} \hookrightarrow D^n \mid n \geq 0 \}$, $\mathcal{J} = \{ 0 \to D^n \mid n > 0 \}$

gives the model category structure on $\mathcal{M} = \mathrm{Ch}$ with

$\mathcal{W} =$ q-iso's, $\mathcal{F} =$ surj's in pos dim's

$\mathcal{C} = {}^\square(\mathcal{I}^\square) =$ retracts of relative $\mathcal{I}$-cell complexes

- attaching a cell: $\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & C \\ \downarrow & \ulcorner & \downarrow \\ D^n & \longrightarrow & D \end{array}$ ⟶

$$\begin{array}{c} \vdots \\ \partial\downarrow \\ C_{n+1} \\ \partial\downarrow \\ C_n \oplus R \\ \partial\downarrow \quad \downarrow 1 \\ C_{n-1} \oplus R \\ \partial\downarrow \\ \vdots \end{array} \qquad \begin{array}{c} \vdots \\ \partial\downarrow \\ C_{n+1} \\ \partial\downarrow \\ C_n \oplus R \\ \partial\downarrow \quad \swarrow f \\ C_{n-1} \\ \partial\downarrow \\ \vdots \end{array}$$

f-glue; arbitrary with $\partial \circ f = 0$

- attaching more cells
  = more copies of $R$ possibly in varying dimensions
- transfinite composition
  = injective map such that the cokernel consists of free modules

  $\Rightarrow$ easy, $\Leftarrow$ uses that our chain complexes are non-negatively graded – build like a CW-cx

- retracts
  = injective map such that the cokernel consists of proj. modules

${}^\square(\mathcal{J}^\square)$ simpler: relative $\mathcal{J}$-cell complexes are inclusions $C \longrightarrow C \oplus \bigoplus_\alpha D^{n_\alpha} \longrightarrow \bigoplus_\alpha D^{n_\alpha}$ with cokernel composed of free modules and contractible $\Rightarrow$ retracts will be inclusions with cokernel composed of proj. modules and still contractible (= projective in Ch)

Remains: $\mathcal{J}$-cell $\subseteq \mathcal{W}$, but this is clear from the description.

Variations.

- Unbounded chain complexes, $\mathcal{F} = $ surj's, $\mathcal{I}$ & $\mathcal{J}$ similar, but cofibrations not so nice
- Bounded below chain complexes, $\mathcal{F} = $ surj's, $\mathcal{I}$ & $\mathcal{J}$ similar, nice description of cofibrations, but only finitely bicomplete.

Homotopy. The path object of $D$ is always

$$\mathrm{Tot}\big(\overset{0}{D \oplus D} \xrightarrow[{[1,1]}]{} \overset{-1}{D}\big) = \mathrm{Hom}\big(\underbrace{\overset{1}{R} \xrightarrow[(-1,1)]{} \overset{0}{R \oplus R}}_{I}, D\big) = \mathrm{Path}\, D$$

$R[0] \xrightarrow[\sim]{i_-} I \xleftarrow[\sim]{i_+} R[0]$, $I \xrightarrow{\sim} R[0]$, $R[0] \xrightarrow{1} R[0]$ yields $D \xleftarrow{\sim} \mathrm{Path}\, D \xrightarrow{\sim} D$, $\mathrm{Path}\, D \xleftarrow{\sim} D$

and right htpy w.r.t. this path object is the usual chain homotopy. Since $Ch_c = $ cx's of proj's, $Ch_f = Ch$, Whitehead theorem says that q-iso between cx's of proj's is a htpy equiv.

Derived functors. Let $F: \text{Mod-}R \to \text{Mod-}S$ be a right exact functor. It induces a functor $F: Ch_R \to Ch_S$ that clearly preserves weak equivalences between cofibrant objects (since by Whitehead theorem these are just homotopy equivalences) and we thus obtain a total left derived functor $\mathbb{L}F(C) = F(C^c)$. In particular, for $C = A[0]$, we have $C^c = P \xrightarrow{\sim} A[0]$ — a projective resolution, and $\mathbb{L}F(A[0]) = FP$, whose homology is $H_n\, \mathbb{L}F(A[0]) = L_n F(A)$.

$\rightsquigarrow$ this gives the left derived functors in a compact way — as an object of $\mathrm{Ho}(Ch_S) \xrightarrow{H_n} \text{Mod-}S$

---

## $\mathcal{M} = \mathrm{Top}$

$\mathcal{W} = $ weak homotopy equivalences

$\mathcal{F} = $ Serre fibrations $= \{0 \times D^n \hookrightarrow I \times D^n\}^{\boxslash}$ ($\swarrow \mathcal{J}$)

$\mathcal{W} \cap \mathcal{F} = $ Serre fibrations with weakly contractible fibres

$$\begin{array}{ccc} S^{n-1} & \to & F \\ \downarrow & \nearrow & \\ D^n & & \end{array}$$

again equivalent to (but more complicated — the problem is homotoped into a fibre)

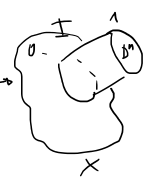
$$\begin{array}{ccc} S^{n-1} & \to & X \\ \downarrow & \nearrow & \downarrow \\ D^n & \to & Y \end{array}$$

$\rightsquigarrow \mathcal{I} = \{S^{n-1} \hookrightarrow D^n \mid n \geq 0\}$, $\mathcal{W} \cap \mathcal{F} = {}^{\boxslash}(\mathcal{I}^{\boxslash})$

$\nabla$ domains of $\mathcal{I}$ & $\mathcal{J}$ are not small but SOA still works

Remains: ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq \mathcal{W}$

"$\subseteq$": a relative $\mathcal{J}$-cell complex is obtained by attaching →




$\rightsquigarrow$ get a deformation retraction

$\Rightarrow$ the inclusion is w.h.e. $\square$

**Derived functors.** The functor $A/Top \to Top$, $(A \to X) \mapsto X/A$ gives a total left derived functor $(A \xrightarrow{f} X) \longmapsto (A \to Cyl\, A +_f X) \longmapsto (Cyl\, A +_f X)/A = Cone\, A +_f X$ (mapping cone)

↑ not cofibrant repl. in this model structure but in a different one (sufficient) (would be cofibrant if $A, X$ were cofibrant)

$X/A$ "cofibre" $Cone\, A +_f X$ "homotopy cofibre"

$\mathcal{M} = sSet = [\Delta^{op}, Set]$

$\mathcal{W}$ = w.h.e. complicated, many equivalent formulations, e.g. maps $f: K \to L$ inducing iso $f^*: [L, X] \to [K, X]$ $\forall X$ Kan cx. (fibrant obj.)

$\mathcal{F} \overset{def}{=}$ Kan fibrations $= \{\Lambda_k \Delta^n \hookrightarrow \Delta^n \mid n \geq 1, k \in \{0, \dots, n\}\}^{\square}$

$= \mathcal{J}^{\square}$ ; ${}^{\square}(\mathcal{J}^{\square})$ = anodyne extensions $= \mathcal{W} \cap \mathcal{C}$

$\mathcal{W} \cap \mathcal{F} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}^{\square} = \mathcal{I}^{\square}$ ; ${}^{\square}(\mathcal{I}^{\square})$ = $\mathcal{I}$-cell = monos = $\mathcal{C}$

There is an adjunction

$|\cdot| : sSet \rightleftarrows Top : S$ ($\perp$)

that clearly takes $|\mathcal{I}_{sSet}| = \mathcal{I}_{Top}$, $|\mathcal{J}_{sSet}| = \mathcal{J}_{Top}$ and preserves colimits $\Rightarrow$ Quillen adjunction

**Theorem.** this is a Quillen equivalence, so that

$|\cdot| : Ho(sSet) \rightleftarrows Ho(Top) : S$ ($\simeq$)

↳ no need to derive, everything in Top fibrant, sSet cofibrant

In particular, $X \in Top \Rightarrow |SX| \xrightarrow{\sim} X$ a functorial CW-replacement

$K \in sSet \Rightarrow K \to S|K|$ a functorial Kan-replacement

# Transfer of the model structure

$F : \mathcal{N} \rightleftarrows \mathcal{M} : G$ and assume that $\mathcal{N}$ has a model structure

Define $\mathcal{W}, \mathcal{F}$ in $\mathcal{M}$ as $G^{-1}\mathcal{W}$, $G^{-1}\mathcal{F}$. Then

$$G^{-1}\mathcal{W} \cap G^{-1}\mathcal{F} = G^{-1}(\mathcal{W} \cap \mathcal{F}) = (F\mathcal{I})^{\boxslash}$$
$$G^{-1}\mathcal{F} = (F\mathcal{J})^{\boxslash}$$

$\Rightarrow$ we get a model category provided that

- $F\mathcal{I}, F\mathcal{J}$ have small domains
- $F\mathcal{J}\text{-cell} \subseteq G^{-1}\mathcal{W}$

**Theorem.** Suppose that $\mathcal{M}$ possesses a functorial fibrant replacement and that there is a functorial path object on $\mathcal{M}_f$ $\leftarrow$ automatic in the enriched context ($\mathrm{Path}\, M = \{\mathrm{Cyl}\, S, M\}$, cylinder on the monoidal unit = "interval")

Then the assumption $F\mathcal{J}\text{-cell} \subseteq G^{-1}\mathcal{W}$ is satisfied.

**Proof.** $F\mathcal{J}\text{-cell} \boxslash G^{-1}\mathcal{F}$

$$F\mathcal{J}\text{-cell} \ni k\downarrow \quad \begin{array}{ccc} M & \xrightarrow{i_M} & M^f \\ \downarrow & \nearrow^{\ell} & \downarrow \\ N & \longrightarrow & 1 \end{array} \quad \Rightarrow \quad \begin{array}{ccc} M & \xrightarrow{i_M} & M^f \\ k\downarrow & \overset{\text{comm}}{\nearrow^{\ell}}\ ? & \downarrow k^f \\ N & \xrightarrow[i_N]{} & N^f \end{array} \qquad M \longmapsto \begin{array}{c} M \xrightarrow{\sim} M^f \\ \downarrow j \\ \mathrm{Path}\, M^f \\ \downarrow q \\ M^f \times M^f \end{array}$$

Similarly:

$$\begin{array}{ccc} M & \xrightarrow{j\, i_N k} & \mathrm{Path}\, N^f \\ k\downarrow & \nearrow & \downarrow q \\ N & \xrightarrow[(i_N, k^f \ell)]{} & N^f \times N^f \end{array} \quad \Rightarrow \quad i_N \sim k^f \ell \text{ so that the lower triangle commutes up to homotopy}$$

Now apply $G$ that preserves path objects and homotopies:

$$\mathcal{M}: \begin{array}{ccc} M & \longrightarrow & M^f \\ \downarrow & \overset{\text{comm}}{\nearrow}{}_{\text{htpy}} & \downarrow \\ N & \longrightarrow & N^f \end{array} \rightsquigarrow \mathcal{N}: \begin{array}{ccc} GM & \xrightarrow{\sim} & GM^f \\ \downarrow & \overset{\text{comm}}{\nearrow}{}_{\text{htpy}} & \downarrow \\ GN & \xrightarrow[\sim]{} & GN^f \end{array} \overset{\gamma}{\rightsquigarrow} \mathrm{Ho}(\mathcal{N}): \begin{array}{ccc} GM & \xrightarrow{\cong} & GM^f \\ \downarrow & \nearrow & \downarrow \\ GN & \xrightarrow[\cong]{} & GN^f \end{array}$$

$$\Downarrow$$

$$M \xrightarrow{\sim} N \quad \Leftarrow \quad GM \xrightarrow{\sim} GN \quad \Leftarrow \quad GM \xrightarrow{\cong} GN \qquad \square$$

**Examples.**

- $F : \mathrm{sSet} \rightleftarrows \mathrm{sAb} : G$ satisfies the assumption since any simplicial group is fibrant (as a simplicial set, i.e. its $G$-image is fibrant) $\Rightarrow$ can take $M^f = M$.
- more generally for any variety of algebras $\mathcal{C}$

$F : sSet \rightleftarrows s\mathcal{C} : G$

there is a fibrant replacement $Ex^\infty$ on sSet that preserves finite limits (it is a filtered colimit of right adjoints) so that it gives a functor

$Ex^\infty : s\mathcal{C} \longrightarrow s\mathcal{C}$

and gives the desired functorial fibrant replacement on $s\mathcal{C}$.

- given $\mathcal{M}$ cofibrantly generated and $\mathcal{A}$ a small category

$$F : [\mathrm{ob}\,\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows [\mathcal{A}^{op}, \mathcal{M}] : G$$

$[\mathrm{ob}\,\mathcal{A}^{op}, \mathcal{M}] \cong \prod_{\mathrm{ob}\,\mathcal{A}} \mathcal{M}$ ← a model category with all $\mathcal{W}, \mathcal{C}, \mathcal{F}$ objectwise

i.e. $\mathcal{I} = \{ i_A \mid A \in \mathcal{A}, i \in \mathcal{I} \}$

$i_A$ = product of
- $i : K \to L$ at object a
- $1 : 0 \to 0$ at object $\neq a$

$$\Rightarrow \quad F i_A = i \cdot \mathcal{A}(-, A) : K \cdot \mathcal{A}(-, A) \longrightarrow L \cdot \mathcal{A}(-, A)$$

with components: $K \cdot \mathcal{A}(B, A) \longrightarrow L \cdot \mathcal{A}(B, A)$

$$\sum_{B \to A} K \xrightarrow{\Sigma i} \sum_{B \to A} L$$

The assumption is satisfied — SOA gives a functorial fibrant replacement.
(Or simply $F\mathcal{J}$-cell has component at $B$ a relative cell complexes generated from $\sum_{B \to A} K \xrightarrow{\Sigma j} \sum_{B \to A} L$ for various $A \in \mathcal{A}$, $j \in \mathcal{J}$ $\Rightarrow$ it lies in $\mathcal{J}$-cell $\subseteq \mathcal{W}$)