Quillen functors – reminder

A Quillen adjunction is an adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$, $F \dashv G$ between model categories s.t.

- F pres. cofibrations $\Leftrightarrow$ G pres. triv. fibrations
- F pres. triv. cofibrations $\Leftrightarrow$ G pres. fibrations

(enough: fibrations between fibrant objects)

Definition. A Quillen adjunction $F \dashv G$ is said to be a Quillen equivalence if $\mathbb{L}F \dashv \mathbb{R}G$ is an (adjoint) equivalence of categories. This happens iff the derived unit and counit are isomorphisms (on htpy categories), i.e. iff

$\mathbb{L}F$ induced by FQ
$\mathbb{R}G$ induced by GR

- $\forall A \in \mathcal{M}_c : \eta' : A \xrightarrow{\eta} GFA \xrightarrow{G r F} GRFA$ is a w.e. (derived unit)
- $\forall X \in \mathcal{N}_f : \varepsilon' : FQGX \xrightarrow{F q G} FGX \xrightarrow{\varepsilon} X$ is a w.e. (derived counit)

Later: Quillen bifunctors, monoidal model categories, enriched model categories

$F : \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \mathcal{N}$
$\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$

Small object argument

Definition. We say that $A \in \mathcal{M}$ is $\varkappa$-small (a cardinal) if, for all $\varkappa$-filtered limit ordinals λ, $\mathcal{M}(A,-)$ preserves λ-indexed colimits

$$\operatorname*{colim}_{\alpha<\lambda} \mathcal{M}(A, M_\alpha) \xrightarrow{\cong} \mathcal{M}(A, \operatorname*{colim}_{\alpha<\lambda} M_\alpha)$$

Ex. $\lambda = \varkappa^+$
$\alpha < \lambda \iff |\alpha| < \varkappa^+ \iff |\alpha| \le \varkappa$

This means that any subset of cardinality $< \varkappa$ has an upper bound (i.e. a supremum in this case)

E.g. $\aleph_0$-filtered ordinal = limit ordinal

$f\colon A \to \operatorname*{colim}_{\alpha<\lambda} M_\alpha$
$\forall a \in A\ \exists \alpha_a : f(a) \in M_{\alpha_a}$
$\exists \alpha : f(a) \in M_\alpha$ — up. bound α_a

Example. A set. A is $\varkappa$-small $\iff |A| < \varkappa$.

Any $\varkappa$-presentable object is $\varkappa$-small $\Rightarrow$ in a l.p. cat any object is $\varkappa$-small for some $\varkappa$.

Complication: In Top, the compact Hausdorff spaces are not quite $\aleph_0$-small but the condition holds if the chain $(M_\alpha)_{\alpha<\lambda}$ consists of closed T_1-inclusions ($f\colon X \to Y$ cl. ind., all pts in $Y \setminus f(X)$ closed)
└ all that will be needed

X cl, cl, Y

Construction. Let I be a set of maps with small domains.

Let $f\colon M \to N$ be arbitrary.

$M \to M_\lambda \to N$, $\in I^\square$ — we want

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & M \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

probably does not exist but we may adjoin to M a solution:

X', X, Y

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & M \arrow[d] \arrow[dd, bend left, "f"] \\ B \arrow[r] \arrow[dr] & M' \arrow[d, dashed] \arrow[ul, phantom, "\ulcorner"] \\ & N \end{tikzcd}$$

Now take all of them:

$\square_s = \begin{tikzcd} A_s \arrow[r] \arrow[d, "i_s"'] & M \arrow[d, "f"] \\ B_s \arrow[r] & N \end{tikzcd}$ with $i_s \in I$ indexed by $s \in S$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "1"] \arrow[d, "n\times"'] & \mathbb{Z} \arrow[d, hook] \\ \mathbb{Z} \arrow[r, dashed] & \mathbb{Z}.\frac{1}{n} \end{tikzcd}$$

$\mathbb{Z}.\frac{1}{n^2}$

$$\begin{tikzcd} A \arrow[r, "\mathrm{in}_s"] \arrow[d, "i"'] & \sum_{s\in S} A_s \arrow[r] \arrow[d, "\sum i_s"'] & M_0 \arrow[d] \arrow[dd, bend left, "f"] \\ B \arrow[r, "\mathrm{in}_s"'] & \sum_{s\in S} B_s \arrow[r] \arrow[dr] & M_1 \arrow[d, dashed] \arrow[ul, phantom, "\ulcorner"] \\ & & N \end{tikzcd}$$

$= M$

$\Rightarrow$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & M_0 \arrow[d] \\ & M_1 \arrow[d] \\ B \arrow[ur, dashed] \arrow[r] & N \end{tikzcd}$$

$/\!/ \ \square_s$

since any such square is one of the squares $\square_s$ and then the diagonal is the restriction of the can. map to B_s

Proceed inductively $\rightsquigarrow$ factor $M_1 \to N$ as $M_1 \to M_2 \to N$ ...
taking $M_\beta = \operatorname*{colim}_{\alpha<\beta} M_\alpha$ for α limit. When do we stop? If the domains of all $i \in I$ are $\varkappa$-small, we stop at any $\varkappa$-filtered limit ordinal λ. ? λ exists; e.g. $\varkappa = \aleph_0$ --- $\lambda = \omega_0$

$M = M_0 \to M_\lambda \to N$, f

domains of all $i \in I$ are κ-small ...

limit ordinal λ. $\quad$! λ exists ; e.g. $\kappa = \aleph_0$ --- $\lambda = \omega_0$

$$M = M_0 \to M_\lambda \to N \qquad (M_0 \xrightarrow{f} N)$$

Theorem. The map $M_0 \to M_\lambda$ is a relative $\mathcal{I}$-cell complex (is built from $\mathcal{I}$ by coproducts, pushouts and transfinite compositions) and the map $M_\lambda \to N$ lies in $\mathcal{I}^\square$.

Proof.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & M_\lambda = \operatorname{colim}_{\alpha<\lambda} M_\alpha \arrow[d] \\ B \arrow[r] & N \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & M_\alpha \arrow[r] & M_{\alpha+1} \arrow[r] & M_\lambda \arrow[d] \\ B \arrow[urr, dashed] \arrow[rrr] & & & N \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r] \arrow[dd] & M_\alpha \arrow[d] \\ & M_{\alpha+1} \arrow[d] \\ & M_\lambda \arrow[d] \\ B \arrow[uur, dashed] \arrow[r] & N \end{tikzcd} \quad \square$$

Corollary. The set $\mathcal{I}$ (maps with small domains) generates a weak factorization system $(\mathcal{L}, \mathcal{R}) = ({}^\square(\mathcal{I}^\square), \mathcal{I}^\square)$, i.e. $\mathcal{L} \square \mathcal{R}$, $\mathcal{M} = \mathcal{R} \circ \mathcal{L}$.
Moreover ${}^\square(\mathcal{I}^\square)$ consists precisely of retracts of relative $\mathcal{I}$-cell cxs.

Proof. $\mathcal{L} \square \mathcal{R}$ by definition, $\mathcal{M} = \mathcal{R} \circ \mathcal{L}$ by the theorem. Let $f \in {}^\square(\mathcal{I}^\square)$ and factor it as in the theorem:

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "i"', "\mathcal{I}\text{-cell} \ni"] & & \cdot \\ & \cdot \arrow[ur, "p \in \mathcal{I}^\square"'] & \end{tikzcd} \qquad f \square p \xRightarrow{\text{retract argument}} f \text{ is a retract of } i. \quad \square$$

This gives a way of constructing examples of model categories

- $\mathcal{M}$ complete & cocomplete
  - $\mathcal{W}$ closed under retracts & 2-out-of-3
  - $\mathcal{F}$ a class of fibrations
  - $\left.\begin{array}{l} \mathcal{W} \cap \mathcal{F} = \mathcal{I}^\square \\ \mathcal{F} = \mathcal{J}^\square \end{array}\right\}$ both $\mathcal{I}$ & $\mathcal{J}$ sets of maps with small domains $\longrightarrow$ this defines $\mathcal{C} = {}^\square(\mathcal{W} \cap \mathcal{F}) = {}^\square(\mathcal{I}^\square)$ = retracts of relative $\mathcal{I}$-cell cxs

What needs to be proved? $\quad$ We get w.f.s.'s

$$({}^\square(\mathcal{I}^\square), \mathcal{I}^\square) = (\mathcal{C}, \mathcal{W} \cap \mathcal{F})$$
$$(\underline{{}^\square(\mathcal{J}^\square)}, \mathcal{J}^\square) \stackrel{?}{=} (\underline{\mathcal{W} \cap \mathcal{C}}, \mathcal{F})$$

Theorem. Assume as above that $\mathcal{M}$ is bicomplete, $\mathcal{W}$ closed under retracts and 2-out-of-3 and $\mathcal{W} \cap \mathcal{F} = \mathcal{I}^\square$, $\mathcal{F} = \mathcal{J}^\square$ for some sets $\mathcal{I}, \mathcal{J}$ of maps with small domains.
Then $\mathcal{M}$ is a model category iff $\underbrace{{}^\square(\mathcal{J}^\square)}_{} \subseteq \mathcal{W}$.
We say that $\mathcal{M}$ is cofibrantly generated. $\quad$ $\mathcal{J}$-cell sufficient

Recall:

MC1 $\quad$ (finite) limits and colimits exist in $\mathcal{M}$

MC2 $\quad$ 2-out-of-3 for $\mathcal{W}$: $\begin{tikzcd} \cdot \arrow[rr, "f \circ g"] \arrow[dr, "g"'] & & \cdot \\ & \cdot \arrow[ur, "f"'] & \end{tikzcd}$ 2 of these in $\mathcal{W}$ $\Rightarrow$ so is 3[rd]

MC3 $\quad$ All $\mathcal{W}, \mathcal{C}, \mathcal{F}$ are closed under retracts

MC4 $\quad$ $\mathcal{C} \square (\mathcal{W} \cap \mathcal{F})$, $\quad (\mathcal{W} \cap \mathcal{C}) \square \mathcal{F}$

MC5 $\quad$ $\mathcal{M} = (\mathcal{W} \cap \mathcal{F}) \circ \mathcal{C}$, $\quad \mathcal{M} = \mathcal{F} \circ (\mathcal{W} \cap \mathcal{C})$

$\Big\}$ $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ & $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$ form the so-called weak factorization systems

Proof. We have $\underset{``\mathcal{J}^{\square}"}{\mathcal{F}} \supseteq \underset{``\mathcal{I}^{\square}"}{\mathcal{W} \cap \mathcal{F}} \Rightarrow {}^{\square}(\mathcal{J}^{\square}) \subseteq {}^{\square}(\mathcal{I}^{\square})$, so that in fact ${}^{\square}(\mathcal{J}^{\square}) \subseteq \mathcal{W} \cap {}^{\square}(\mathcal{I}^{\square})$. We have seen that

$${}^{\square}(\mathcal{J}^{\square}) = \mathcal{W} \cap {}^{\square}(\mathcal{I}^{\square})$$

is sufficient, so we proceed to show "$\supseteq$".

Thus, let $f \in \mathcal{W} \cap {}^{\square}(\mathcal{I}^{\square})$ and factor it using SOA w.r.t. $\mathcal{J}$:

$$\begin{tikzcd} \cdot \arrow[rr, "f \,\in\, \mathcal{W} \cap {}^{\square}(\mathcal{I}^{\square})"] \arrow[dr, "i"'] & & \cdot \\ & \cdot \arrow[ur, "p \,\in\, \mathcal{J}^{\square} = \mathcal{F}"'] & \end{tikzcd}$$

$i \in \mathcal{J}$-cell

$i \in \mathcal{J}\text{-cell} \subseteq \mathcal{W} \Rightarrow p \in \mathcal{W}$ by 2-out-of-3

$\Rightarrow p \in \mathcal{W} \cap \mathcal{F} = \mathcal{I}^{\square}$

$\Rightarrow f \,\square\, p \xRightarrow[\text{argument}]{\text{retract}} f$ is a retract of i

$\Rightarrow f \in {}^{\square}(\mathcal{J}^{\square})$ $\square$

Examples

One of the simplest examples is $\mathcal{M} = \mathrm{Cat}$

$\mathcal{W}$ = equivalences of categories

$\mathcal{F}$ = isofibrations = $F : \mathcal{C} \to \mathcal{D}$ s.t. given $c \in \mathcal{C}$ and an iso $Fc \cong d$ a lift $c \cong c'$ exists
$Fc' = d$

$$\begin{tikzcd} \{\cdot\} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ \{\cdot \xrightarrow{\cong} \cdot\} \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd} \qquad \rightsquigarrow \mathcal{J} = \{\{0\} \hookrightarrow \{0 \cong 1\}\}$$

$F : \mathcal{C} \to \mathcal{D}$ $\quad \forall d\ \exists c : Fc \cong d$

$\mathcal{W} \cap \mathcal{F}$ = surjective equivalences of categories
↳ ess. surj. on objects; if iso-fib ⇒ surj.

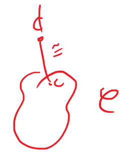
$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ \{\cdot\} \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

surjectivity on objects

surjectivity on maps

injectivity on maps

$$\begin{tikzcd} \{\cdot \quad \cdot\} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ \{\cdot \to \cdot\} \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \{\cdot \rightrightarrows \cdot\} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ \{\cdot \to \cdot\} \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$



$\mathcal{C}$

$$\begin{tikzcd} 0 \arrow[r, "c"] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \cong 1 \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

$\forall c', c \in \mathcal{C}$

$\mathcal{C}(c', c) \xrightarrow{F} \mathcal{D}(Fc', Fc)$

$\rightsquigarrow \mathcal{I} = \{\emptyset \hookrightarrow \{0\}, \{0 \quad 1\} \hookrightarrow \{0 \to 1\}, \{0 \rightrightarrows 1\} \longrightarrow \{0 \to 1\}\}$

Clearly: $\mathcal{J}$-cell = injective equivalences $= {}^{\square}(\mathcal{J}^{\square})$
$\mathcal{I}$-cell = functors injective on objects $= {}^{\square}(\mathcal{I})^{\square}$ $\Big\}$ $\Rightarrow {}^{\square}(\mathcal{J}^{\square}) = \mathcal{W} \cap {}^{\square}(\mathcal{I}^{\square})$

$\mathrm{Cyl}\, \mathcal{C} = (0 \cong 1) \times \mathcal{C}$ $\longrightarrow$ htpy = nat. iso. $\qquad {}^{\square}(\mathcal{J}^{\square}) \subseteq \mathcal{W}$

$\mathcal{M} = \mathrm{Ch}$, say non-negatively graded chain complexes of (right) R-modules
$\cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$

$\mathcal{W}$ = quasi-isomorphisms

$\mathcal{F}$ = maps that are surjective in positive dimensions $\qquad \mathrm{Ch}_f = \mathrm{Ch}$

Define for $n > 0$: $D^n = (\cdots \longrightarrow 0 \longrightarrow \overset{n}{R} \xrightarrow{1} \overset{n-1}{R} \longrightarrow 0 \longrightarrow \cdots)$

$\mathrm{Ch}(D^n, C) \cong C_n$ so that

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & C \arrow[d, "f"] \\ D^n \arrow[r] \arrow[ur, dashed] & D \end{tikzcd} \qquad \begin{matrix} C_n \\ \downarrow \text{surj.} \\ D_n \end{matrix} \iff f \in \mathcal{F}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \overset{n}{R} \arrow[r, "1"] \arrow[d] & \overset{n-1}{R} \arrow[r] \arrow[d] & 0 \arrow[d] \\ C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & C_{n-2} \end{tikzcd}$$

$\mathcal{W} \cap \mathcal{F}$ = surjective quasi-iso's; $\quad f : C \to D$ induces

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d, "\text{surj}"] & H_0 C \arrow[r] \arrow[d, "\cong"] & 0 \\ D_1 \arrow[r] & D_0 \arrow[r] & H_0 D \arrow[r] & 0 \end{tikzcd}$$

= surjective + kernel acyclic
zero homology

Define for $n > 0$: $S^{n-1} = (\cdots \longrightarrow 0 \longrightarrow \overset{n}{0} \longrightarrow \overset{n-1}{R} \longrightarrow 0 \longrightarrow \cdots) \subseteq D^n$

$\mathrm{Ch}(S^{n-1}, K) \cong Z_{n-1} K$ $\quad$ $(n-1)$-cycles of K

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d] & K \\ \cdot \arrow[ur, dashed, "c"'] & \end{tikzcd} \iff H_{n-1} K = 0 \qquad \begin{matrix} 0 z = 0 \\ z = \partial c \end{matrix}$$

S^{n-2} $\quad S^{n-1}$ $\quad S^{n-1} \longrightarrow D^n$ $\qquad$ Remark. $n = 0$

$Ch(S^{n-1}, K) \cong Z_{n-1} K$ $(n-1)$-cycles of K

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & K \\ D^n \arrow[ur, dashed, "c"'] \end{tikzcd} \quad \partial c = z \quad \Leftrightarrow H_{n-1} K = 0$$

$$\begin{tikzcd} Ch(D^n, K) \arrow[d, "i^*"] \arrow[r, phantom, "\cong"] & K_n \arrow[d, "\partial"] \\ Ch(S^{n-1}, K) \arrow[r, phantom, "\cong"] & Z_{n-1} K \end{tikzcd}$$

$$\begin{tikzcd} S^{n-2} \arrow[r] \arrow[d, "0"'] & D^{n-1} \arrow[d, "pr"] \\ 0 \arrow[r] & S^{n-1} \end{tikzcd} \qquad \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d, "1"] \\ S^{n-1} \arrow[r] & D^n \end{tikzcd}$$

Remark. $n = 0$

$S^{-1} \longrightarrow D^0$ $\quad Z_{-1} C = 0$

$= 0 \longrightarrow S^0$ $\quad C_0 = Z_0 C$

$\Rightarrow \quad \mathcal{I}^\square \subseteq \mathcal{J}^\square =$ surj's in pos. dim's

$\rightsquigarrow \quad \mathcal{I}^\square \subseteq \mathcal{W} \cap \mathcal{J}^\square$ since

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "0"'] \arrow[ur, dashed] & D \end{tikzcd} \equiv \begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & \ker f \\ D^n \arrow[ur, dashed] \end{tikzcd} \Leftrightarrow H_{n-1} \ker f = 0$$

For the opposite direction: let $f \in \mathcal{W} \cap \mathcal{J}^\square$, i.e. f is a surj. quasi-iso

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "d"'] \arrow[ur, dashed] & D \end{tikzcd} - \begin{tikzcd} S^{n-1} \arrow[r, "\partial c"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "d"'] \arrow[ur, "c"] & D \end{tikzcd} = \begin{tikzcd} S^{n-1} \arrow[r, "z - \partial c"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "0"'] \arrow[ur, dashed] & D \end{tikzcd}$$

has a solution since $\ker f$ is acyclic

by surjectivity, c exists

Summary. $\mathcal{I} = \{ S^{n-1} \hookrightarrow D^n \mid n \geq 0 \}$, $\mathcal{J} = \{ 0 \to D^n \mid n > 0 \}$

gives the model category structure on $\mathcal{M} = Ch$ with

$\mathcal{W} =$ q-iso's, $\mathcal{F} =$ surj's in pos dim's

$\mathcal{C} = {}^\square(\mathcal{I}^\square) =$ retracts of relative $\mathcal{I}$-cell complexes

- attaching a cell:

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & C \arrow[d] \\ D^n \arrow[r] & D \end{tikzcd}$$

$$\begin{tikzcd} \vdots \arrow[d, "\partial"] \\ C_{n+1} \arrow[d, "\partial"] \\ C_n \oplus R \arrow[d, "\partial"] \arrow[dr, phantom, "1"] \\ C_{n-1} \oplus R \arrow[d, "\partial"] \\ \vdots \end{tikzcd} \qquad f\text{-glue} \qquad \begin{tikzcd} \vdots \arrow[d, "\partial"] \\ C_{n+1} \arrow[d, "\partial"] \\ C_n \oplus R \arrow[d, "\partial"] \\ C_{n-1} \arrow[d, "\partial"] \\ \vdots \end{tikzcd}$$

arbitrary f with $\partial \circ f = 0$

- attaching more cells
  = more copies of R
  possibly in varying dimensions
- transfinite composition
  = injective map such that the cokernel consists of free modules

$\Rightarrow$ easy, $\Leftarrow$ uses that our chain complexes are non-negatively graded - build like a CW-cx

- retracts
  = injective map such that the cokernel consists of proj. modules

${}^\square(\mathcal{J}^\square)$ simpler: relative $\mathcal{J}$-cell complexes are inclusions $C \longrightarrow C \oplus \bigoplus_\alpha D^{n_\alpha} \longrightarrow \bigoplus_\alpha D^{n_\alpha}$ with cokernel composed of free modules and contractible $\Rightarrow$ retracts will be inclusions with cokernel composed of proj. modules and still contractible (= projective in Ch)

$$\begin{tikzcd} \bigoplus 0 \arrow[r] \arrow[d] & C \arrow[d] \\ \bigoplus_\alpha D^{n_\alpha} \arrow[r] & C \oplus \bigoplus D^{n_\alpha} \end{tikzcd}$$

Remains: $\mathcal{J}$-cell $\subseteq \mathcal{W}$, but this is clear from the description.

Variations.

Remains: $\mathcal{J}$-cell $\subseteq \mathcal{W}$, ...

Variations.

- Unbounded chain complexes, $\mathcal{F}$ = surj's, $\mathcal{I}$ & $\mathcal{J}$ similar, but cofibrations not so nice
- Bounded below chain complexes, $\mathcal{F}$ = surj's, $\mathcal{I}$ & $\mathcal{J}$ similar, nice description? of cofibrations, but only finitely bicomplete.

Homotopy. The path object of D is always

$$\mathrm{Tot}\,(\overset{0}{D} \oplus D \xrightarrow[{[1,1]}]{} \overset{-1}{D}) = \mathrm{Hom}\,(\underbrace{\overset{1}{R} \xrightarrow[(-1,1)]{} \overset{0}{R \oplus R}}_{I}, D) = \mathrm{Path}\,D$$

$$\begin{tikzcd} R[0] \arrow[r, "i_-", "\sim"'] \arrow[dr, "1"'] & I \arrow[d, "\sim"] & R[0] \arrow[l, "i_+"', "\sim"] \arrow[dl, "1"] \\ & R[0] & \end{tikzcd} \quad \text{yields} \quad \begin{tikzcd} D \arrow[r] \arrow[dr, "1"'] & \mathrm{Path}\,D \arrow[d, "\sim"] & D \arrow[l] \arrow[dl, "1"] \\ & D & \end{tikzcd}$$

and right htpy w.r.t. this path object is the usual chain homotopy. Since Ch_c = cx's of proj's, $Ch_f = Ch$, Whitehead theorem says that q-iso between cx's of proj's is a htpy equiv.

Derived functors. Let $F\colon \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}S$ be a right exact functor. It induces a functor $F\colon Ch_R \to Ch_S$ that clearly preserves weak equivalences between cofibrant objects (since by Whitehead theorem these are just homotopy equivalences) and we thus obtain a total left derived functor $\mathbb{L}F(C) = F(C^c)$. In particular, for $C = A[0]$, we have $C^c = P \xrightarrow{\sim} A[0]$ — a projective resolution, and $\mathbb{L}F(A[0]) = FP$, whose homology is $H_n\,\mathbb{L}F(A[0]) = L_nF(A)$.

$\rightsquigarrow$ this gives the left derived functors in a compact way — as an object of $\mathrm{Ho}(Ch_S) \xrightarrow{H_n} \mathrm{Mod}\text{-}S$

$\mathcal{M}$ = Top

$\mathcal{W}$ = weak homotopy equivalences

$\mathcal{F}$ = Serre fibrations $= \{0 \times D^n \hookrightarrow I \times D^n\}^{\boxslash}$ ($\mathcal{J}$)

$\mathcal{W} \cap \mathcal{F}$ = Serre fibrations with weakly contractible fibres

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & F \\ D^n \arrow[ur, dashed] & \end{tikzcd}$$

again equivalent to (but more complicated — the problem is homotoped into a fibre)

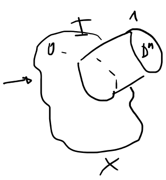
$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$\rightsquigarrow \mathcal{I} = \{S^{n-1} \hookrightarrow D^n \mid n \geq 0\}$, $\mathcal{W} \cap \mathcal{F} = {}^{\boxslash}(\mathcal{I}^{\boxslash})$

∇ domains of $\mathcal{I}$ & $\mathcal{J}$ are not small but SOA still works

Remains: ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq \mathcal{W}$

"$\subseteq$": a relative $\mathcal{J}$-cell complex is obtained by attaching



$\rightsquigarrow$ get a deformation retraction

...

" $\in$: a relative ...

$\rightsquigarrow$ get a deformation retraction

$\Rightarrow$ the inclusion is w.h.e. $\square$

Derived functors. The functor $A/Top \to Top$ gives a total left
$(A \to X) \mapsto X/A$

derived functor $(A \xrightarrow{f} X) \longmapsto (A \to Cyl\, A +_f X) \longmapsto (Cyl\, A +_f X)/A = Cone\, A +_f X$ mapping cone

$\llcorner$ not cofibrant repl. in this model structure but in a different one (sufficient) (would be cofibrant if A, X were cofibrant)

X/A "cofibre" $\quad Cone\, A +_f X$ " homotopy cofibre"

$\overline{M} = \text{sSet} = [\Delta^{op}, Set]$

$\mathcal{W}$ = w.h.e. complicated, many equivalent formulations,

e.g. maps $f : K \to L$ inducing iso $f^* : [L, X] \to [K, X]$ $\forall X$ Kan cx. (fibrant obj.)

$\mathcal{F} \overset{def}{=}$ Kan fibrations $= \{\Lambda_k \Delta^n \hookrightarrow \Delta^n \mid n \geq 1,\ k \in \{0, \ldots, n\}\}^{\boxslash}$

$= \mathcal{J}^{\boxslash}$; ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ = anodyne extensions $= \mathcal{W} \cap \mathcal{C}$

$\mathcal{W} \cap \mathcal{F} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}^{\boxslash} = \mathcal{I}^{\boxslash}$; ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ = $\mathcal{I}$-cell = monos = $\mathcal{C}$

There is an adjunction

$|\cdot| : \text{sSet} \rightleftarrows Top : S$ ($\perp$)

that clearly takes $|\mathcal{I}_{sSet}| = \mathcal{I}_{Top}$, $|\mathcal{J}_{sSet}| = \mathcal{J}_{Top}$ and preserves colimits $\Rightarrow$ Quillen adjunction

Theorem. This is a Quillen equivalence, so that

$|\cdot| : Ho(sSet) \underset{\simeq}{\rightleftarrows} Ho(Top) : S$

$\llcorner$ no need to derive, everything in Top fibrant, sSet cofibrant

In particular, $X \in Top \Rightarrow |SX| \xrightarrow{\sim} X$ a functorial CW-replacement

$K \in sSet \Rightarrow K \to S|K|$ a functorial Kan-replament

Transfer of the model structure

$F: \mathcal{N} \rightleftarrows \mathcal{M} : G$ and assume that $\mathcal{N}$ has a model structure

Define $\mathcal{W}, \mathcal{F}$ in $\mathcal{M}$ as $G^{-1}\mathcal{W}$, $G^{-1}\mathcal{F}$. Then

$$G^{-1}\mathcal{W} \cap G^{-1}\mathcal{F} = G^{-1}(\mathcal{W} \cap \mathcal{F}) = (F\mathcal{I})^{\boxslash}$$
$$G^{-1}\mathcal{F} = (F\mathcal{J})^{\boxslash}$$

$\Rightarrow$ we get a model category provided that

- $F\mathcal{I}, F\mathcal{J}$ have small domains
- $F\mathcal{J}$-cell $\subseteq G^{-1}\mathcal{W}$

Theorem. Suppose that $\mathcal{M}$ possesses a functorial fibrant replacement and that there is a functorial path object on $\mathcal{M}_f$. Then the assumption $F\mathcal{J}\text{-cell} \subseteq G^{-1}\mathcal{W}$ is satisfied.

automatic in the enriched context

$\left(\text{Path}\, M = \{\underbrace{\text{Cyl}\, S}_{\substack{\text{cylinder on the monoidal unit} \\ = \text{"interval"}}}, M\}\right)$

Proof. $F\mathcal{J}\text{-cell} \boxslash G^{-1}\mathcal{F}$

$$\begin{tikzcd} M \arrow[r, "i_M"] \arrow[d, "k"'] & M^f \arrow[d] \\ N \arrow[r] \arrow[ur, dashed, "\ell"] & 1 \end{tikzcd} \quad F\mathcal{J}\text{-cell} \ni k \qquad \Rightarrow \qquad \begin{tikzcd} M \arrow[r, "i_M"] \arrow[d, "k"'] & M^f \arrow[d, "k^f"] \\ N \arrow[r, "i_N"'] \arrow[ur, "\ell", "?"'] & N^f \end{tikzcd} \ \text{comm}$$

$$M \longmapsto \begin{tikzcd} M \arrow[r, "\sim"] & M^f \arrow[d, "j"] \\ & \text{Path}\, M^f \arrow[d, "q"] \\ & M^f \times M^f \end{tikzcd}$$

Similarly:

$$\begin{tikzcd} M \arrow[r, "j i_N k"] \arrow[d, "k"'] & \text{Path}\, N^f \arrow[d, "q"] \\ N \arrow[r, "{(i_N, k^f \ell)}"'] \arrow[ur, dashed] & N^f \times N^f \end{tikzcd} \quad \Rightarrow \quad i_N \sim k^f \ell$$

so that the lower triangle commutes up to homotopy

Now apply G that preserves path objects and homotopies:

$$\mathcal{M}: \begin{tikzcd} M \arrow[r] \arrow[d] & M^f \arrow[d] \\ N \arrow[r] \arrow[ur, "\text{htpy}"] & N^f \end{tikzcd} \ \text{comm} \quad \rightsquigarrow \quad \mathcal{N}: \begin{tikzcd} GM \arrow[r, "\sim"] \arrow[d] & GM^f \arrow[d] \\ GN \arrow[r, "\sim"'] \arrow[ur, "\text{htpy}"] & GN^f \end{tikzcd} \ \text{comm} \quad \overset{\gamma}{\rightsquigarrow} \quad \mathcal{H}o(\mathcal{N}): \begin{tikzcd} GM \arrow[r, "\cong"] \arrow[d] & GM^f \arrow[d] \\ GN \arrow[r, "\cong"'] \arrow[ur] & GN^f \end{tikzcd}$$

$$\Downarrow$$

$$M \xrightarrow{\sim} N \quad \Longleftarrow \quad GM \xrightarrow{\sim} GN \quad \Longleftarrow \quad GM \xrightarrow{\cong} GN \qquad \square$$

Examples.

- $F: \text{sSet} \rightleftarrows \text{sAb} : G$ satisfies the assumption since any simplicial group is fibrant (as a simplicial set, i.e. its G-image is fibrant) $\Rightarrow$ can take $M^f = M$.
- more generally for any variety of algebras $\mathcal{C}$

$F : sSet \rightleftarrows s\mathcal{C} : G$

there is a fibrant replacement Ex^∞ on sSet that preserves finite limits (it is a filtered colimit of right adjoints) so that it gives a functor

$Ex^\infty : s\mathcal{C} \longrightarrow s\mathcal{C}$

and gives the desired functorial fibrant replacement on $s\mathcal{C}$.

- given $\mathcal{M}$ cofibrantly generated and $\mathcal{A}$ a small category

$F : [\mathrm{ob}\,\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows [\mathcal{A}^{op}, \mathcal{M}] : G$

$\quad\quad \cong$

$\prod_{\mathrm{ob}\,\mathcal{A}} \mathcal{M} \longleftarrow$ a model category with all $\mathcal{W}, \mathcal{C}, \mathcal{F}$ objectwise

i.e. $\mathcal{I} = \{ i_A \mid A \in \mathcal{A}, i \in \mathcal{I} \}$

$\llcorner$ product of $\bullet$ $i : K \to L$ at object a

$\bullet$ $1 : 0 \to 0$ at object $\neq a$

$\Rightarrow F i_A = i \cdot \mathcal{A}(-, A) : K \cdot \mathcal{A}(-, A) \longrightarrow L \cdot \mathcal{A}(-, A)$

with components: $K \cdot \mathcal{A}(B, A) \longrightarrow L \cdot \mathcal{A}(B, A)$

$$\sum_{B \to A} K \xrightarrow[\Sigma i]{} \sum_{B \to A} L$$

The assumption is satisfied — SOA gives a functorial fibrant replacement.
(Or simply FJ-cell has component at B a relative cell complexes generated from $\sum_{B \to A} K \xrightarrow[\Sigma j]{} \sum_{B \to A} L$ for various $A \in \mathcal{A}$, $j \in \mathcal{J}$ $\Rightarrow$ it lies in $\mathcal{J}$-cell $\subseteq \mathcal{W}$)