# Examples

$\mathcal{M} = Ch$, say non-negatively graded chain complexes of (right) $R$-modules $\cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$

$\mathcal{W} =$ quasi-isomorphisms

$\mathcal{F} =$ maps that are surjective in positive dimensions $Ch_f = Ch$

Define for $n > 0$: $D^n = ( \cdots \to \overset{n}{0} \to R \xrightarrow{1} \overset{n-1}{R} \to 0 \to \cdots)$

$Ch(D^n, C) \cong C_n$ so that

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & C \arrow[d, "f"] \\ D^n \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

$$\begin{tikzcd} C_n \arrow[d, "surj."] \\ D_n \end{tikzcd}$$

$\Longleftrightarrow f \in \mathcal{F}$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \overset{n}{R} \arrow[r, "1"] \arrow[d] & \overset{n-1}{R} \arrow[r] \arrow[d] & 0 \arrow[d] \\ C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & C_{n-2} \end{tikzcd}$$

$\mathcal{W} \cap \mathcal{F} =$ surjective quasi-iso's; $f: C \to D$ induces

$=$ surjective + kernel acyclic

zero homology

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d, "surj"] & H_0 C \arrow[r] \arrow[d, "\cong"] & 0 \\ D_1 \arrow[r] & D_0 \arrow[r] & H_0 D \arrow[r] & 0 \end{tikzcd}$$

Define for $n > 0$: $S^{n-1} = ( \cdots \to 0 \to \overset{n}{0} \to \overset{n-1}{R} \to 0 \to \cdots ) \subseteq D^n$

$Ch(S^{n-1}, K) \cong Z_{n-1} K$ $(n-1)$-cycles of $K$

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & K \\ D^n \arrow[ur, dashed, "c"] \end{tikzcd}$$

$\partial z = 0$

$z = \partial c$

$\Longleftrightarrow H_{n-1} K = 0$

$$\begin{tikzcd} Ch(D^n, K) \arrow[r, phantom, "\cong"] \arrow[d, "i^*"] & K_n \arrow[d, "\partial"] \\ Ch(S^{n-1}, K) \arrow[r, phantom, "\cong"] & Z_{n-1} K \end{tikzcd}$$

$$\begin{tikzcd} S^{n-2} \arrow[r] \arrow[d, "0"] & D^{n-1} \arrow[d, "pr"] & S^{n-1} \arrow[r] \arrow[d, "1"] & D^n \arrow[d, "1"] \\ 0 \arrow[r] & S^{n-1} & S^{n-1} \arrow[r] & D^n \end{tikzcd}$$

**Remark.** $n = 0$

$S^{-1} \longrightarrow D^0$ $\quad Z_{-1} C = 0$

$= 0 \longrightarrow S^0$ $\quad C_0 = Z_0 C$

$\Rightarrow \mathcal{I}^\square \subseteq \mathcal{J}^\square =$ surj's in pos. dim's

$\rightsquigarrow \mathcal{I}^\square \subseteq \mathcal{W} \cap \mathcal{J}^\square$ since

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "0"] \arrow[ur, dashed] & D \end{tikzcd} \equiv \begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & \ker f \\ D^n \arrow[ur, dashed] \end{tikzcd}$$

$\Leftarrow H_{n-1} \ker f = 0$

For the opposite direction: let $f \in \mathcal{W} \cap \mathcal{J}^\square$, i.e. $f$ is a surj. quasi-iso

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "d"] \arrow[ur, dashed] & D \end{tikzcd} - \begin{tikzcd} S^{n-1} \arrow[r, "\partial c"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "d"] \arrow[ur, "c"] & D \end{tikzcd} = \begin{tikzcd} S^{n-1} \arrow[r, "z - \partial c"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "0"] \arrow[ur, dashed] & D \end{tikzcd}$$

has a solution since $\ker f$ is acyclic

by surjectivity, $c$ exists

Summary. $\mathcal{I} = \{ S^{n-1} \hookrightarrow D^n \mid n \geq 0 \}$, $\mathcal{J} = \{ 0 \to D^n \mid n > 0 \}$

gives the model category structure on $\mathcal{M} = Ch$ with

$\mathcal{W} =$ q-iso's, $\mathcal{F} =$ surj's in pos dim's

$\mathcal{C} = {}^\square(\mathcal{I}^\square) =$ retracts of relative $\mathcal{I}$-cell complexes

$$\begin{tikzcd} S^{n} \\ 0 \arrow[d] \\ \overset{n}{0} \arrow[d] \\ \overset{n-1}{R} \arrow[d] \\ 0 \end{tikzcd} \quad \hookrightarrow \quad \begin{tikzcd} D^n \\ 0 \arrow[d] \\ R \arrow[d, "1"] \\ R \arrow[d] \\ 0 \end{tikzcd}$$

- attaching a cell: $$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & C \arrow[d] \\ D^n \arrow[r] & D \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

- attaching more cells
  $=$ more copies of $R$

$\rightsquigarrow$

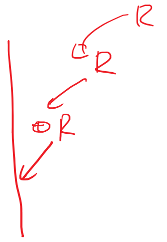
$$\begin{tikzcd} \vdots \arrow[d, "\partial"] & \\ C_{n+1} \arrow[d, "\partial"] & \\ C_n \oplus R \arrow[d, "\partial"] & \arrow[d, "1"] \\ {} & {} \end{tikzcd} \qquad \begin{tikzcd} \vdots \arrow[d, "\partial"] & \\ C_{n+1} \arrow[d, "\partial"] & \\ C_n \oplus R \arrow[d, "\partial"] & \arrow[dl, "f"] \\ {} & \end{tikzcd}$$

arbitrary with $\partial \circ f = 0$

- attaching more cells
  = more copies of R possibly in varying dimensions →

$$\begin{tikzcd} C_n \arrow[d, "\partial"'] & \oplus & R \arrow[d, "1"] \\ C_{n-1} \arrow[d, "\partial"'] & \oplus & R \arrow[l, bend left, "f\text{--glue}"] \\ \vdots & & \end{tikzcd} \qquad \begin{tikzcd} C_n \arrow[d, "\partial"'] & \oplus & R \arrow[dll, "f"] \\ C_{n-1} \arrow[d, "\partial"'] & & \\ \vdots & & \end{tikzcd}$$

arbitrary with $\partial \circ f = 0$



- transfinite composition
  = injective map such that the cokernel consists of free modules

  ⇒ easy, ⇐ uses that our chain complexes are non-negatively graded
  - build like a CW-cx

- retracts
  = injective map such that the cokernel consists of proj. modules

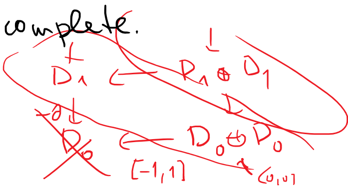
$^{\square}(\mathcal{J}^{\square})$ simpler: relative $\mathcal{J}$-cell complexes are inclusions $C \longrightarrow C \oplus \bigoplus_{\alpha} D^{n_\alpha} \longrightarrow \bigoplus_{\alpha} D^{n_\alpha}$ with cokernel composed of free modules and contractible ⇒ retracts will be inclusions with cokernel composed of proj. modules and still contractible (= projective in Ch)

$$\begin{tikzcd} \bigoplus 0 \arrow[r] \arrow[d] & C \arrow[d] \\ \bigoplus_{\alpha} D^{n_\alpha} \arrow[r] & C \oplus \bigoplus D^{n_\alpha} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Remains: $\mathcal{J}$-cell $\subseteq \mathcal{W}$, but this is clear from the description.

## Variations.

- Unbounded chain complexes, $\mathcal{F}$ = surj's, $\mathcal{I}$ & $\mathcal{J}$ similar, but cofibrations not so nice
- Bounded below chain complexes, $\mathcal{F}$ = surj's, $\mathcal{I}$ & $\mathcal{J}$ similar, nice description of cofibrations, but only finitely bicomplete.



## Homotopy.

The path object of D is always

$$\mathrm{Tot}\,(\overset{0}{D \oplus D} \xrightarrow[{[-1,1]}]{} \overset{-1}{D}) = \mathrm{Hom}\,(\underbrace{\overset{1}{R} \xrightarrow[(-1,1)]{} \overset{0}{R \oplus R}}_{I}, D) = \mathrm{Path}\ D$$

$$(\mathrm{Path}\ D)_n = D_n \oplus D_n \oplus D_{n+1}$$

$$\begin{tikzcd} R[0] \arrow[r, "i_-", "\sim"'] \arrow[dr, "1"'] & I \arrow[d, "\sim"'] & R[0] \arrow[l, "i_+"', "\sim"] \arrow[dl, "1"] \\ & R[0] & \end{tikzcd} \quad \text{yields} \quad \begin{tikzcd} D & \mathrm{Path}\ D \arrow[l, "\sim"'] \arrow[r, "\sim"] & D \\ & D \arrow[ul, "1"] \arrow[u, "\sim"] \arrow[ur, "1"'] & \end{tikzcd}$$

and right htpy w.r.t. this path object is the usual chain homotopy. Since $Ch_c$ = cx's of proj's, $Ch_f = Ch$, Whitehead theorem says that q-iso between cx's of proj's is a htpy equiv.

## Derived functors.

Let $F\colon \text{Mod-}R \to \text{Mod-}S$ be a right exact functor.

$F(0) = 0$ additive enough

It induces a functor $F\colon Ch_R \to Ch_S$ that clearly preserves weak equivalences between cofibrant objects (since by Whitehead theorem these are just homotopy equivalences) and we thus obtain a total left derived functor $\mathbb{L}F(C) = F(C^c)$. In particular, for $C = A[0]$, we have $C^c = P \xrightarrow{\sim} A[0]$ — a projective resolution, and $\mathbb{L}F(A[0]) = FP$, whose homology is $H_n\,\mathbb{L}F(A[0]) = L_nF(A)$.

⇝ this gives the left derived functors in a compact way — as an object of $Ho(Ch_S) \xrightarrow{H_n} \text{Mod-}S$

$\Psi$

$$\begin{tikzcd} Ho(Ch_R) \arrow[r, "\mathbb{L}F"] \arrow[drr, bend right] & Ho(Ch_S) \arrow[dr, "\mathbb{L}G"] & \\ & & Ho(Ch_T) \end{tikzcd}$$

⇝ this gives the left derived functors in a compa

– as an object of $\mathrm{Ho}(\mathrm{Ch}_S) \xrightarrow{H_n} \mathrm{Mod}\text{-}S$

$$\begin{tikzcd}
\mathrm{Ho}(\mathrm{Ch}_R) \arrow[r, "\mathbb{L}F"] \arrow[rr, bend right, "\mathbb{L}(G\circ F)"'] & \mathrm{Ho}(\mathrm{Ch}_S) \arrow[r, "\mathbb{L}G"] & \mathrm{Ho}(\mathrm{Ch}_T)
\end{tikzcd}$$

$$\mathbb{L}F(A[0]) \longmapsto L_nF(A)$$

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## $\mathcal{M} = \mathrm{Top}$

$\mathcal{W}$ = weak homotopy equivalences

$\mathcal{F}$ = Serre fibrations $= \{0\times D^n \hookrightarrow I\times D^n\}^{\boxslash}$ ↙ $\mathcal{J}$

$$\begin{tikzcd}
0\times D^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
I\times D^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

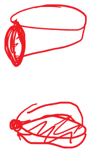
$\mathcal{W}\cap\mathcal{F}$ = Serre fibrations with weakly contractible fibres

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & F \\
D^n \arrow[ur, dashed] &
\end{tikzcd}$$

again equivalent to (but more complicated – the problem is homotoped into a fibre )

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] \arrow[r, bend right, "2"'] & X \arrow[d] \\
D^n \arrow[r] \arrow[ur, dashed, "\text{im in } F?"] \arrow[r, bend right=40, "2"'] & Y
\end{tikzcd}$$

$*$

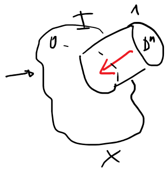


⇝ $\mathcal{I} = \{S^{n-1} \hookrightarrow D^n \mid n \geq 0\}$, $\mathcal{W}\cap\mathcal{F} = \mathcal{I}^{\boxslash}$

$\overset{!}{\nabla}$ domains of $\mathcal{I}$ & $\mathcal{J}$ are not small but SOA still works

Remains: ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq \mathcal{W}$

"$\subseteq$": a relative $\mathcal{J}$-cell complex is obtained by attaching →



⇝ get a deformation retraction

⇒ the inclusion is w.h.e. $(A \xrightarrow{in} A+Y) \longleftarrow\!\!| \; Y$

$\square$

**Derived functors.** The functor $A/\mathrm{Top} \to \mathrm{Top}$ gives a total left

$(A \to X) \longmapsto X/A$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & * \arrow[d] \\
X \arrow[r] & X/A
\end{tikzcd}$$

mapping cone

derived functor $(A \xrightarrow{f} X) \longmapsto (A \to \mathrm{Cyl}\, A +_f X) \longmapsto (\mathrm{Cyl}\, A +_f X)/A = \mathrm{Cone}\, A +_f X$

↑ not cofibrant repl. in this model structure but in a different one (sufficient) (would be cofibrant if $A, X$ were cofibrant)

$$\begin{tikzcd}
A+A \arrow[r, "f"] \arrow[d] & A+X \arrow[d] & A \arrow[l] \\
\mathrm{Cyl}\,A \arrow[r] & \mathrm{Cyl}\,f &
\end{tikzcd}$$

A cof   X cof

$X/A$ "cofibre"   $\mathrm{Cone}\, A +_f X$ "homotopy cofibre"

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## $\mathcal{M}$ = sSet $= [\Delta^{op}, \mathrm{Set}]$

$\mathcal{W}$ = w.h.e. complicated, many equivalent formulations,

e.g. maps $f: K \to L$ inducing iso $f^*: [L, X] \to [K, X]$ $\forall X$ Kan cx.

(Kan cx. = fibrant obj.)

$\mathcal{F} \overset{\text{def}}{=}$ Kan fibrations $= \{\Lambda_k\Delta^n \hookrightarrow \Delta^n \mid n > 1, k \in \{0, \ldots, n\}\}^{\boxslash}$

$= \mathcal{J}^{\boxslash}$; ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ = anodyne extensions $= \mathcal{W}\cap\mathcal{C}$

$\mathcal{W}\cap\mathcal{F} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}^{\boxslash} = \mathcal{I}^{\boxslash}$; ${}^{\boxslash}(\mathcal{I}^{\boxslash}) = \mathcal{I}$-cell = monos = $\mathcal{C}$

There is an adjunction

$$|\cdot| : \mathrm{sSet} \underset{\perp}{\rightleftarrows} \mathrm{Top} : S$$

that clearly takes $|\mathcal{I}_{\mathrm{sSet}}| = \mathcal{I}_{\mathrm{Top}}$, $|\mathcal{J}_{\mathrm{sSet}}| = \mathcal{J}_{\mathrm{Top}}$ and preserves colimits ⇒ Quillen adjunction ← F pres cof. ⇐ FI consists of cof's

**Theorem**. This is a Quillen equivalence, so that

$|\cdot| : Ho(sSet) \overset{\simeq}{\rightleftarrows} Ho(Top) : S$

└ no need to derive, everything in Top fibrant, sSet cofibrant

In particular, $X \in Top \Rightarrow |SX| \xrightarrow{\sim} X$ a functorial CW-replacement

$K \in sSet \Rightarrow K \xrightarrow{\sim} S|K|$ a functorial Kan-replament

# Transfer of the model structure

$F : \mathcal{N} \rightleftarrows \mathcal{M} : G$ and assume that $\mathcal{N}$ has a cofibrantly generated model structure

Define $\mathcal{W}, \mathcal{F}$ in $\mathcal{M}$ as $G^{-1}\mathcal{W}$, $G^{-1}\mathcal{F}$. Then

$$G^{-1}\mathcal{W} \cap G^{-1}\mathcal{F} = G^{-1}(\mathcal{W}\cap\mathcal{F}) = (F\mathcal{I})^{\boxslash}$$

$$G^{-1}\mathcal{F} = (F\mathcal{J})^{\boxslash}$$

G preserves and reflects fib's and w.e's

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "Fi"'] & X \arrow[d, "p"] \\ FB \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad (\Leftrightarrow) \quad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & GX \arrow[d, "Gp"] \\ B \arrow[r] \arrow[ur] & GY \end{tikzcd}$$

$\Rightarrow$ we get a model category provided that

- $F\mathcal{I}, F\mathcal{J}$ have small domains
- $F\mathcal{J}$-cell $\subseteq G^{-1}\mathcal{W}$

Quillen; Goerss-Jardine

**Theorem.** Suppose that $\mathcal{M}$ possesses a functorial fibrant replacement for which a path object exists. Then the assumption $F\mathcal{J}$-cell $\subseteq G^{-1}\mathcal{W}$ is satisfied.

automatic in the enriched context

$\left(\text{Path}\, M = \{\underbrace{\text{Cyl}\, S}_{\substack{\text{cylinder on the monoidal unit} \\ = \text{"interval"}}}, M\}\right)$

**Proof.** $F\mathcal{J}$-cell $\boxslash$ $G^{-1}\mathcal{F}$

$$F\mathcal{J}\text{-cell} \ni k \quad \begin{tikzcd} M \arrow[r, "i_M"] \arrow[d, "k"'] & M^f \arrow[d] \\ N \arrow[r] \arrow[ur, dashed, "\ell"] & 1 \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} M \arrow[r, "i_M"] \arrow[d, "k"'] \arrow[dr, phantom, "\text{comm}"] & M^f \arrow[d, "k^f"] \\ N \arrow[r, "i_N"'] \arrow[ur, "\ell", "?"'] & N^f \end{tikzcd}$$

functorial $M \xrightarrow{\sim} M^f$

i.e. $M^f$ functor, $\downarrow$ nat. transf.

Similarly:

$$\begin{tikzcd} M \arrow[r, "j i_N k"] \arrow[d, "k"'] & \text{Path}\, N^f \arrow[d, "q"] \\ N \arrow[r, "{(i_N, k^f\ell)}"'] \arrow[ur, dashed] & N^f \times N^f \end{tikzcd} \quad \Rightarrow \quad i_N \sim k^f\ell$$

so that the lower triangle above commutes up to right homotopy

Now apply G that preserves path objects and right homotopies:

$$\overset{\mathcal{M}}{\begin{tikzcd} M \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\text{comm}"] & M^f \arrow[d] \\ N \arrow[r, "\sim"'] \arrow[ur, "\text{htpy}"'] & N^f \end{tikzcd}} \rightsquigarrow \overset{\mathcal{N}}{\begin{tikzcd} GM \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\text{comm}"] & GM^f \arrow[d] \\ GN \arrow[r, "\sim"'] \arrow[ur, "\text{htpy}"'] & GN^f \end{tikzcd}} \overset{\gamma}{\rightsquigarrow} \overset{\text{Ho}(\mathcal{N})}{\begin{tikzcd} GM \arrow[r, "\cong"] \arrow[d] & GM^f \arrow[d] \\ GN \arrow[r, "\cong"'] \arrow[ur] & GN^f \end{tikzcd}}$$

$$\Downarrow$$

$$M \xrightarrow{\sim} N \quad \Leftarrow \quad GM \xrightarrow{\sim} GN \quad \Leftarrow \quad GM \xrightarrow{\cong} GN \qquad \square$$

**Examples.**

- $F : \text{sSet} \rightleftarrows \text{sAb} : G$ satisfies the assumption since any simplicial group is fibrant (as a simplicial set, i.e. its G-image is fibrant) $\Rightarrow$ can take $M^f = M$.
- more generally for any variety of algebras $\mathcal{C}$

  $F : \text{sSet} \rightleftarrows s\mathcal{C} : G$

  there is a fibrant replacement $Ex^\infty$ on sSet that preserves finite limits (it is a filtered colimit of right adjoints) so that it gives a functor

  $(Ex^\infty X)^n \leftarrow \text{colim}(X^n \to ExX^n \to Ex^2X^n \to \cdots)$

  $X \to ExX \to \cdots$

  $Ex^\infty : s\mathcal{C} \to s\mathcal{C}$

  and gives the desired functorial fibrant replacement on $s\mathcal{C}$.
- given $\mathcal{M}$ cofibrantly generated and $\mathcal{A}$ a small category

  $F : [\text{ob}\,\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows [\mathcal{A}^{op}, \mathcal{M}] : G$

  $\text{ob}\,A \hookrightarrow A$

  $[\text{ob}\,\mathcal{A}^{op}, \mathcal{M}] \cong \prod_{\text{ob}\,\mathcal{A}} \mathcal{M} \leftarrow$ a model category with all $\mathcal{W}, \mathcal{C}, \mathcal{F}$ objectwise

  $(M_0, M_1) \longleftarrow\!\shortmid (M_0 \to M_1)$

  $$\begin{tikzcd} K_0 \arrow[d] & 0 \arrow[d] \\ L_0 & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[d] & K_1 \arrow[d] \\ 0 & L_1 \end{tikzcd}$$

  i.e. $\mathcal{I} = \{ i_A \mid A \in \mathcal{A}, i \in \mathcal{I} \}$

  $\llcorner$ product of
  - $i : K \to L$ at object $A$
  - $1 : 0 \to 0$ at object $\neq A$

$F(K_0, 0) = K_0 \xrightarrow{1} K_0$

$F(0, K_1) = 0 \to K_1$

$[\mathcal{A}^{op}, \mathcal{M}](K \cdot \mathcal{A}(-, A), X)$

$\Rightarrow F i_A = i \cdot \mathcal{A}(-, A) : K \cdot \mathcal{A}(-, A) \to L \cdot \mathcal{A}(-, A)$

$K \cdot \mathcal{A}(B, A) \to L \cdot \mathcal{A}(B, A)$

$F(0, K_1) = 0 \to K_1$

$[\mathcal{A}^{op}, \mathcal{M}](K \cdot \mathcal{A}(-,A), X)$
$\cong \mathcal{M}(K, X(A))$

- $1 : 0 \to 0$ at object $\neq A$

$$\Rightarrow F i_A = i \cdot \mathcal{A}(-,A) : K \cdot \mathcal{A}(-,A) \longrightarrow L \cdot \mathcal{A}(-,A)$$

with components:

$$\begin{tikzcd} K \cdot \mathcal{A}(B,A) \arrow[r] \arrow[d, equal] & L \cdot \mathcal{A}(B,A) \arrow[d, equal] \\ \sum_{B \to A} K \arrow[r, "\Sigma i"'] & \sum_{B \to A} L \end{tikzcd}$$

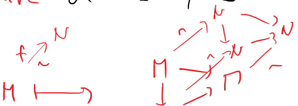
The assumption is satisfied — SOA gives a functorial fibrant replacement. (Or simply $F\mathcal{J}$-cell has component at $B$ a relative cell complexes generated from $\sum_{B \to A} K \xrightarrow[\Sigma j]{} \sum_{B \to A} L$ for various $A \in \mathcal{A}$, $j \in \mathcal{J}$ $\Rightarrow$ it lies in $\mathcal{J}$-cell $\subseteq \mathcal{W}$)

We get the so called projective model structure on $[\mathcal{A}^{op}, \mathcal{M}]$.

The adjunction:

$$\mathrm{colim} : [\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows \mathcal{M} : \Delta$$

$\Delta : M \longmapsto$



is a Quillen adjunction — $\Delta$ preserves fibrations and weak equivalences.

$$\mathbb{L}\,\mathrm{colim} : \mathrm{Ho}\,[\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows \mathrm{Ho}\,\mathcal{M} : \mathbb{R}\Delta$$

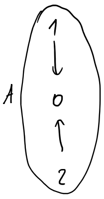
$\llcorner$ essentially $\Delta$

$\mathbb{R}\Delta(M) = \Delta(M^f)$
$\uparrow \sim$
$\Delta(M) = \Delta(M)$

$\mathrm{colim} \circ Q : [\mathcal{A}^{op}, \mathcal{M}] \longrightarrow \mathcal{M}$ the "homotopy colimit"; concrete models using concrete cofibrant replacements

Remark. colim does not preserve general weak equiv's, only w.e. between cofibrant objects (i.e. cofib. repl. is necessary):

$\mathcal{A}$



$$\begin{tikzcd} D^n \arrow[r, "\sim"] & * \\ S^{n-1} \arrow[u] \arrow[d] \arrow[r, "="] & S^{n-1} \arrow[u] \arrow[d] \\ D^n \arrow[r, "\sim"] & * \end{tikzcd}$$

cofibrant not cofibrant

$\rightsquigarrow$

$$\begin{tikzcd} S^n \arrow[r, "/" marking] & * \end{tikzcd}$$

$\uparrow$ the homotopy colimit of the second diag.

$$\begin{tikzcd} & \text{cofibrant} \arrow[d, "/" marking] & & \\ & 0 = 0 \cdot \mathcal{A}(-,0) \arrow[r, "\text{gen.cof.}"] & S^{n-1} \cdot \mathcal{A}(-,0) \arrow[r, "L"'] & D \\ & & S^{n-1} \cdot \mathcal{A}(-,1) \arrow[u] \arrow[r, "\text{gen.cof.}"] & D^n \cdot \mathcal{A}(-,1) \arrow[u] \\ & & + & + \\ & & S^{n-1} \cdot \mathcal{A}(-,2) \arrow[r, "\text{gen.cof.}"'] & D^n \cdot \mathcal{A}(-,2) \end{tikzcd}$$

$$\begin{tikzcd} S^{n-1} \\ S^{n-1} \arrow[u, "1"] \arrow[d, "1"'] \\ S^{n-1} \end{tikzcd}$$

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] & D^n \\ 0 \arrow[u] \arrow[d] \arrow[r, hook] & 0 \arrow[u] \arrow[d] \\ 0 \arrow[r, hook] & 0 \end{tikzcd}$$

Remark. Generally, cofibrations are natural transformations

$$\begin{tikzcd} X \arrow[r, tail] \arrow[rr, tail, bend left] & \cdot \arrow[r, tail] & X' \\ A \arrow[u] \arrow[d] \arrow[rr, tail] & & A' \arrow[u] \arrow[d] \arrow[ul] \arrow[dl] \\ Y \arrow[r, tail] \arrow[rr, tail, bend right] & \cdot \arrow[r, tail] & Y' \end{tikzcd}$$

(all components and the maps from the pushouts are cofibrations

$\Rightarrow$ cofibrant diagrams = all objects cofibrant and both maps cofibrations

# Properness

**Theorem.** Let $\mathcal{M}$ be a model category.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & B \arrow[d, tail] \\ X \arrow[r, "g"'] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

If all objects are cofibrant then

$f$ w.e. $\Rightarrow$ $g$ w.e.

**Definition** We say that $\mathcal{M}$ is left proper if the same holds for all objects $A, B, X, Y$ (not necessarilly cofibrant), i.e. w.e.'s are preserved by pushouts along cofibrations. Dually, $\mathcal{M}$ is right proper if we!s are preserved by pullbacks along fibrations.

**Example.** If all objects of $\mathcal{M}$ are cofibrant $\Rightarrow$ $\mathcal{M}$ left proper

e.g. $\mathcal{M} = \mathrm{Cat}, \mathrm{sSet}$ ; serious example: Top, Ch.

Dually $\mathcal{M} = \mathrm{Cat}, \mathrm{Top}, \mathrm{Ch}$ right proper for free; sSet serious example.

$\mathcal{M}$ left proper $\Rightarrow$ $[\mathcal{A}^{op}, \mathcal{M}]$ left proper (pushouts and w.e. pointwise, cof $\Rightarrow$ ptwise cof)

**Proof.** Clearly holds if $f$ is a trivial cofibration $\xRightarrow[???]{\text{Brown}}$ holds also for w.e.'s between cofibrant objects.

not that simple

$$f_* : A/\mathcal{M} \rightleftarrows B/\mathcal{M} : f^*$$

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dd, bend left, "f^*y"] & & B \arrow[d, "y"] \\ B \arrow[d, "y"'] & & Y \arrow[ll, mapsto] \\ Y & & \end{tikzcd}$$

$\leftarrow$ preserves & reflects fibrations and weak equivalences (they are determined via $\mathrm{cod}: A/\mathcal{M} \to \mathcal{M}$, $(A \to X) \mapsto X$ and $f^*$ commutes with them.)

$$\begin{tikzcd} A \arrow[d, "x"'] & & A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "f_* x"] \\ X \arrow[rr, mapsto] & & X \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$\Rightarrow$ derived unit on cofibrant objects

$$\eta' : 1 \xrightarrow{\eta} f^* f_* \xrightarrow[\sim]{f^* r f_*} f^* R f_*$$

$\uparrow$ serves equally well

Summary:

when $f$ is a trivial cofibration then $\mathbb{L}f_* : \mathrm{Ho}\, A/\mathcal{M} \rightleftarrows \mathrm{Ho}\, B/\mathcal{M} : \mathbb{R}f^*$ is an equivalence ($\eta'$ w.e. $\Rightarrow \varepsilon'$ w.e.)

2-out-of-3: 
$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$$
induces

$$\begin{tikzcd} & \mathrm{Ho}(B/\mathcal{M}) \arrow[dl, shift left] \arrow[dr, shift left] & \\ \mathrm{Ho}(A/\mathcal{M}) \arrow[ur, shift left] \arrow[rr, shift left] & & \mathrm{Ho}(C/\mathcal{M}) \arrow[ll, shift left] \arrow[ul, shift left] \end{tikzcd}$$

two equivalences $\Rightarrow$ so is third

cofibrant i.e. $x$ cofibration $\Rightarrow$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, bend left, "f^* f_* x"] & B \arrow[d, "f_* x"] \\ X \arrow[r, "\eta"'] & Y \end{tikzcd}$$

$\Rightarrow$ if $\eta'$ is a w.e. then so is $\varepsilon'$ by the triangle identity:

$$\begin{tikzcd} \mathbb{R}f^* X \arrow[rr, "1"] \arrow[dr, "\eta' \mathbb{R}f^*"', "\cong"] & & \mathbb{R}f^* X \\ & \mathbb{R}f^* \mathbb{L}f_* \mathbb{R}f^* X \arrow[ur, "\mathbb{R}f^* \varepsilon'"'] & \end{tikzcd}$$

$\mathbb{R}f^* \varepsilon'$ iso $\Downarrow$ $\varepsilon'$ iso

Now we can finally apply Brown's lemma 𝄋

ε' iso

Now we can finally apply Brown's lemma:

$$\begin{tikzcd} & \mathrm{Cyl}\,f \arrow[dr, "p"] & & & & \mathrm{Cyl}\,f \arrow[dr, "p"] & \\ A \arrow[ur, "i"] \arrow[rr, "f"'] & & B \arrow[ul, dashed, "j"'] & \arrow[l] & B \arrow[ur, "j"] \arrow[rr, "1"'] & & B \end{tikzcd}$$

$i_* \dashv i^*$ and $j_* \dashv j^*$ Q.e.

$\Downarrow$

$f_* \dashv f^*$ Q.e. $\quad p_* \dashv p^*$ Q.e. $\square$

Remark. A different and still conceptual proof: let $\mathcal{A} = \{\overset{0}{\cdot} \longrightarrow \overset{1}{\cdot} \longleftarrow \overset{2}{\cdot}\}$ and equip $[\mathcal{A}^{op}, \mathcal{M}]$ with the so-called Reedy model structure (see later) in which a span $M_0 \longleftarrow M_1 \longrightarrow M_2$ is cofibrant iff all $M_0, M_1, M_2$ are cofibrant and the map $M_1 \longrightarrow M_2$ is a cofibration (unlike in the proj. model str. where both maps need to be cofibrations) we still get a Quillen adjunction

$$\mathrm{colim} : [\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows \mathcal{M} : \Delta$$

and thus colim preserves w.e. between cofibrant objects. Apply to

$$\begin{tikzcd} A \arrow[r, "f", "\sim"'] & B \\ A \arrow[u, "1"] \arrow[r, "1"] \arrow[d, tail] & A \arrow[u, "f"'] \arrow[d, tail] \\ X \arrow[r, "1"] & X \end{tikzcd} \quad \overset{\mathrm{colim}}{\rightsquigarrow} \quad X \xrightarrow[\sim]{g} Y$$

# Quillen bifunctors, monoidal and enriched model categories

**Definition.** A left adjoint bifunctor is a bifunctor

$F : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$

that, for each $n \in \mathcal{N}$, yields a left adjoint functor $F(-,n) : \mathcal{M} \to \mathcal{P}$ and, for each $m \in \mathcal{M}$, yields a left adjoint functor $F(m,-) : \mathcal{N} \to \mathcal{P}$

$\rightsquigarrow$ get $G : \mathcal{N}^{op} \times \mathcal{P} \to \mathcal{M}$ s.t. $\mathcal{P}(F(m,n), p) \cong \mathcal{M}(m, G(n,p))$
and $H : \mathcal{M}^{op} \times \mathcal{P} \to \mathcal{N}$ s.t. $\mathcal{P}(F(m,n), p) \cong \mathcal{N}(n, H(m,p))$

We may call $(F,G,H)$ an adjunction of two variables.

**Definition.** A left Quillen bifunctor is a left adjoint bifunctor such that for each cofibration $i : A \rightarrowtail B$ in $\mathcal{M}$ and $j : K \rightarrowtail L$ in $\mathcal{N}$, the map $F_\lrcorner(i,j) \overset{def}{=} F(A,L) +_{F(A,K)} F(B,K) \longrightarrow F(B,L)$ as in

$$\begin{tikzcd}
F(A,K) \arrow[rr] \arrow[dd] & & F(B,K) \arrow[dl] \arrow[dd] \\
& \ulcorner F_\lrcorner(i,j) \arrow[dr] & \\
F(A,L) \arrow[ur] \arrow[rr] & & F(B,L)
\end{tikzcd}$$

is a cofibration that is trivial if at least one of $i,j$ is.

We may write this as:

$$F_\lrcorner(\mathcal{C},\mathcal{C}) \subseteq \mathcal{C}$$
$$F_\lrcorner(\mathcal{W} \cap \mathcal{C}, \mathcal{C}) \subseteq \mathcal{W} \cap \mathcal{C}$$
$$F_\lrcorner(\mathcal{C}, \mathcal{W} \cap \mathcal{C}) \subseteq \mathcal{W} \cap \mathcal{C}$$

**Remark.** If $\mathcal{M}, \mathcal{N}$ are cofibrantly generated, it is enough to check this for the generating cofibrations and trivial cofibrations. This also applied to the case of Quillen functors (easier).

**Remark.** This easily generalizes to $\underline{n}$ variables giving as special cases:

$n=1$: left Quillen functor
$n=2$: left Quillen bifunctor

**Lemma.** $F$ is left Quillen $\Leftrightarrow$ $G$ is right Quillen, i.e.

$$G_\ulcorner(\mathcal{C}, \mathcal{F}) \subseteq \mathcal{F}$$
$$G_\ulcorner(\mathcal{W} \cap \mathcal{C}, \mathcal{F}) \subseteq \mathcal{W} \cap \mathcal{F}$$
$$G_\ulcorner(\mathcal{C}, \mathcal{W} \cap \mathcal{F}) \subseteq \mathcal{W} \cap \mathcal{F}$$

here I decided to denote fibrations in $\mathcal{N}^{op}$ by $\mathcal{C}$ since they are cofibrations in $\mathcal{N}$

**Proof.** This boils down to $F_\lrcorner(i,j) \,\square\, p \Leftrightarrow i \,\square\, G_\ulcorner(j,p)$ that must be checked and is tedious but completely elementary. $\square$

**Definition.** A monoidal model category is a category with

- a closed monoidal structure, i.e. $\otimes$ is a left biadjoint with right adjoints $\hom_r$ and $\hom_\ell$
- a model structure

↳ symmetric for us → $\hom_r = \hom_\ell = \{\,,\,\}$

such that

- the unit of the monoidal structure $S$ is either cofibrant or, more generally, the cofibrant replacement $S^c \xrightarrow{\sim} S$ tensors with all cofibrant objects to w.e.'s
$$S^c \otimes X \xrightarrow{\sim} S \otimes X \cong X$$
$$X \otimes S^c \xrightarrow{\sim} X \otimes S \cong X$$
this does not depend on the choice of the cofibrant replacement in view of
- the tensor product bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ is left Quillen.

Examples. All $\mathrm{Cat}, \underbrace{\mathrm{Ch}, \mathrm{Top}}, \mathrm{sSet}$ are monoidal model categories.
(with cofibrant unit)

$$\left.\begin{array}{l} i_n : S^{n-1} \longrightarrow D^n \\ j_0 : \bullet \longrightarrow I \\ j_n = j_0 \otimes_\lrcorner i_n \end{array}\right\}$$

$i_n \otimes_\lrcorner i_m$ is a cofibration

$\left.\begin{array}{l} j_n \otimes_\lrcorner i_m \\ i_n \otimes_\lrcorner j_m \end{array}\right\}$ are trivial cofibrations

$$\begin{tikzcd} m & R \arrow[d, "(-1)^{m+1}"'] & R \arrow[l, "1"'] \arrow[d, "(-1)^n"] \\ m-1 & R & R \arrow[l, "1"] \\ & n-1 & n \end{tikzcd}$$

A functorial cylinder object for cofibrant objects: pick a cylinder object for the unit $S \in \mathcal{V}_c$: $S + S \rightarrowtail \mathrm{Cyl}\, S \xrightarrow{\sim} S$ and tensor it with $A \in \mathcal{V}_c$ to obtain

↳ a w.e. between cofibrant objects

$$\begin{tikzcd} A \otimes (S+S) \arrow[r, tail] \arrow[d, equal] & A \otimes \mathrm{Cyl}\, S \arrow[r, "\sim"] \arrow[d, equal, "\mathrm{def}"] & A \otimes S \arrow[d, equal] \\ A + A \arrow[r, tail] & \mathrm{Cyl}\, A \arrow[r, "\sim"] & A \end{tikzcd}$$

Dually a path object is $\{\mathrm{Cyl}\, S, X\}$.

Definition. A $\mathcal{V}$-category $\mathcal{M}$ is said to be tensored if

$$\mathcal{V}(V, \mathcal{M}(M,N)) \cong \mathcal{M}(M \otimes V, N) \quad \text{naturally}$$

This gives a functor $\otimes : \mathcal{M}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{M}_0$ (in fact $\mathcal{M} \otimes \mathcal{V} \dashv \mathcal{M}$) that makes $\mathcal{M}_0$ into a "module" over $\mathcal{V}_0$. Dually $\mathcal{M}$ is cotensored if $\mathcal{V}(V, \mathcal{M}(M,N)) \cong \mathcal{M}(M, \{V, N\})$ naturally

This gives a functor $\{\,,\,\} : \mathcal{V}_0^{op} \times \mathcal{M}_0 \longrightarrow \mathcal{M}_0$

Together with the hom-functor $\mathcal{M}(-,-) : \mathcal{M}_0 \times \mathcal{M}_0 \longrightarrow \mathcal{V}_0$ these yield an adjunction of two variables.

If $A \otimes S^c \xrightarrow{\sim} A \otimes S = A$ for $A \in \mathcal{M}_c$ and this adjunction is Quillen, we say that $\mathcal{M}$ is a model $\mathcal{V}$-category.

Again $A \otimes \mathrm{Cyl}\, S$ is a cylinder object for $A \in \mathcal{M}_c$
$\{\mathrm{Cyl}\, S, X\}$ is a path object for $X \in \mathcal{M}_f$.

Left adjoint $\mathcal{V}$-functors preserve these cylinder objects (they are colimits),
right adjoint $\mathcal{V}$-functors preserve these path objects (they are limits).

# Reedy model categories, framings