Properness

Theorem. Let $\mathcal{M}$ be a model category.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & B \arrow[d, tail] \\ X \arrow[r, "g"'] & Y \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

If all objects are cofibrant then f w.e. $\Rightarrow$ g w.e.

Definition. We say that $\mathcal{M}$ is left proper if the same holds for all objects A, B, X, Y (not necessarily cofibrant), i.e. w.e.'s are preserved by pushouts along cofibrations. Dually, $\mathcal{M}$ is right proper if w.e.'s are preserved by pullbacks along fibrations.

Example. If all objects of $\mathcal{M}$ are cofibrant $\Rightarrow$ $\mathcal{M}$ left proper

e.g. $\mathcal{M} = \mathrm{Cat}, \mathrm{sSet}$; serious example: Top, Ch.

Dually $\mathcal{M} = \mathrm{Cat}, \mathrm{Top}, \mathrm{Ch}$ right proper for free; sSet serious example.

$\mathcal{M}$ left proper $\Rightarrow$ $[\mathcal{A}^{op}, \mathcal{M}]$ left proper (pushouts and w.e. pointwise, cof $\Rightarrow$ ptwise cof)

Proof. Clearly holds if f is a trivial cofibration $\overset{\text{Brown}}{\underset{???}{\Longrightarrow}}$ holds also for w.e.'s between cofibrant objects. not that simple

$$f_* : A/\mathcal{M} \rightleftarrows B/\mathcal{M} : f^*$$

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \end{tikzcd} \quad \text{under } A: f^*y,\ f^*z; \qquad \text{under } B: y,\ z$$

$$\left(\begin{array}{c} A \\ \downarrow f \\ B \\ \downarrow y \\ Y \end{array}\right) f^*y \longleftarrow\!\shortmid \begin{array}{c} B \\ \downarrow y \\ Y \end{array}$$

$\longleftarrow$ preserves & reflects fibrations and weak equivalences (they are determined via $\mathrm{cod} : A/\mathcal{M} \to \mathcal{M}$, $(A \to X) \mapsto X$ and f^* commutes with them.)

$$\begin{array}{c} A \\ x\downarrow \\ X \end{array} \longmapsto \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "f_* x"] \\ X \arrow[r] & Y \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$\Rightarrow$ derived unit on cofibrant objects

$$\eta' : 1 \xrightarrow{\eta} f^* f_* \underset{\sim}{\xrightarrow{f^* r f_*}} f^* R f_*$$

$\uparrow$ serves equally well

Summary:

when f is a trivial cofibration then $\mathbb{L}f_* : \mathrm{Ho}\, A/\mathcal{M} \rightleftarrows \mathrm{Ho}\, B/\mathcal{M} : \mathbb{R}f^*$ is an equivalence (η' w.e. $\Rightarrow \varepsilon'$ w.e.)

2-out-of-3: $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$ induces

$$\begin{tikzcd} & \mathrm{Ho}(B/\mathcal{M}) \arrow[dl, shift left] \arrow[dr, shift left] & \\ \mathrm{Ho}(A/\mathcal{M}) \arrow[ur, shift left] \arrow[rr, shift left] & & \mathrm{Ho}(C/\mathcal{M}) \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd}$$

two equivalences $\Rightarrow$ so is third

cofibrant i.e. x cofibration

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "f^* f_* x"', shift right] \arrow[d, "f_* x", shift left] \\ X \arrow[r, "\eta"'] & Y \end{tikzcd}$$

$\Rightarrow$ if η' is a w.e. then so is ε' by the triangle identity:

$$\begin{tikzcd} \mathbb{R}f^* X \arrow[rr, "1"] \arrow[dr, "\eta' \mathbb{R}f^*"', "\cong"] & & \mathbb{R}f^* X \\ & \mathbb{R}f^* \mathbb{L}f_* \mathbb{R}f^* X \arrow[ur, "\mathbb{R}f^* \varepsilon'"'] & \end{tikzcd}$$

$\mathbb{R}f^* \varepsilon'$ iso $\Downarrow$ ε' iso

Now we can finally apply Brown's lemma &

Now we can finally apply Brown's lemma: $\qquad \varepsilon'$ iso

$$
\begin{tikzcd}
 & \mathrm{Cyl}\,f \arrow[dr, "p"] & \\
A \arrow[ur, "i", "\text{triv. cof}"'] \arrow[rr, "f"', "\text{w.e. between cof. obj's}"'] & & B \arrow[ul, dotted, "j"', "\text{triv. cof.}"]
\end{tikzcd}
\longleftarrow
\begin{tikzcd}
 & \mathrm{Cyl}\,f \arrow[dr, "p"] & \\
B \arrow[ur, "j"] \arrow[rr, "1"'] & & B
\end{tikzcd}
$$

$i_* \dashv i^*$ and $j_* \dashv j^*$ Q.e.

$\Downarrow$

$f_* \dashv f^*$ Q.e. $\Leftarrow$ $p_* \dashv p^*$ Q.e. $\square$

Remark. A different and still conceptual proof: let $\mathcal{A} = \{\overset{0}{\cdot} \longrightarrow \overset{1}{\cdot} \longleftarrow \overset{2}{\cdot}\}$ and equip $[\mathcal{A}^{op}, \mathcal{M}]$ with the so-called Reedy model structure (see later) in which a span $M_0 \longleftarrow M_1 \longrightarrow M_2$ is cofibrant iff all M_0, M_1, M_2 are cofibrant and the map $M_1 \longrightarrow M_2$ is a cofibration (unlike in the proj. model str. where both maps need to be cofibrations). We still get a Quillen adjunction

$$\mathrm{colim} : [\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows \mathcal{M} : \Delta$$

and thus colim preserves w.e. between cofibrant objects. Apply to

$$
\begin{tikzcd}
A \arrow[r, "f", "\sim"'] & B \\
A \arrow[u, "1"] \arrow[r, "1"] \arrow[d, tail] & A \arrow[u, "f"'] \arrow[d, tail] \\
X \arrow[r, "1"] & X
\end{tikzcd}
\quad \overset{\mathrm{colim}}{\rightsquigarrow} \quad X \xrightarrow[\sim]{g} Y
$$

Quillen bifunctors, monoidal and enriched model categories

Definition. A *left adjoint bifunctor* is a bifunctor

$$F: \mathcal{M}\times\mathcal{N}\longrightarrow\mathcal{P}$$

that, for each $N\in\mathcal{N}$, yields a left adjoint functor $F(-,N):\mathcal{M}\to\mathcal{P}$
and, for each $M\in\mathcal{M}$, yields a left adjoint functor $F(M,-):\mathcal{N}\to\mathcal{P}$

$A\otimes B\to C$
$A\to \mathrm{Hom}(B,C)$
$-\otimes B \dashv \mathrm{Hom}(B,-)$

$\rightsquigarrow$ get $G:\mathcal{N}^{op}\times\mathcal{P}\to\mathcal{M}$ s.t. $\mathcal{P}(F(M,N),P)\cong\mathcal{M}(M,G(N,P))$
and $H:\mathcal{M}^{op}\times\mathcal{P}\to\mathcal{N}$ s.t. $\mathcal{P}(F(M,N),P)\cong\mathcal{N}(N,H(M,P))$

We may call (F,G,H) an adjunction of two variables.

Definition. A *left Quillen bifunctor* is a left adjoint bifunctor such that for each cofibration $i:A\rightarrowtail B$ in $\mathcal{M}$ and $j:K\rightarrowtail L$ in $\mathcal{N}$, the map $F_\lrcorner(i,j): F(A,L)+_{F(A,K)}F(B,K)\longrightarrow F(B,L)$ as in

pushout corner map

$$\begin{tikzcd}
F(A,K) \arrow[r, "{F(i,1)}"] \arrow[d, "{F(1,j)}"'] & F(B,K) \arrow[d] \arrow[dd, bend left, "{F(1,j)}"] \\
F(A,L) \arrow[r] \arrow[dr, "{F(i,1)}"'] & \bullet \arrow[d, "{F_\lrcorner(i,j)}"] \\
& F(B,L)
\end{tikzcd}$$

$N\xrightarrow{f}N'$

$G(N,P)$ $\quad F(G(N,P),N)\xrightarrow{\varepsilon_N}P$
$\uparrow\exists!$ $\qquad\quad\uparrow\exists!\qquad\qquad\uparrow\varepsilon_{N'}$
$G(N',P)$ $\quad F(G(N',P),N)\xrightarrow{F(1,f)}F(G(N',P),N')$

make $F\to F'$ $\quad G'\to G$

is a cofibration that is trivial if at least one of i,j is.

We may write this as:

$$F_\lrcorner(\mathcal{C},\mathcal{C})\subseteq\mathcal{C}$$
$$F_\lrcorner(\mathcal{W}\cap\mathcal{C},\mathcal{C})\subseteq\mathcal{W}\cap\mathcal{C}$$
$$F_\lrcorner(\mathcal{C},\mathcal{W}\cap\mathcal{C})\subseteq\mathcal{W}\cap\mathcal{C}$$

sSet: $A\subseteq B$ $\quad K\subseteq L$ $\quad F=\times$
$(A\times L)+_{A\times K}(B\times K)\longrightarrow B\times L$
$\times$ is left Q. bifunctor

Remark. If $\mathcal{M},\mathcal{N}$ are cofibrantly generated, it is enough to check this for the generating cofibrations and trivial cofibrations.
This also applied to the case of Quillen functors (easier).

Remark. This easily generalizes to $\underline{n}$ variables giving as special cases:

$n=1$: left Quillen functor
$n=2$: left Quillen bifunctor

G_r

Lemma. F is left Quillen $\Leftrightarrow$ G is right Quillen, i.e.

$$G_r(\mathcal{C},\mathcal{F})\subseteq\mathcal{F}$$
$$G_r(\mathcal{W}\cap\mathcal{C},\mathcal{F})\subseteq\mathcal{W}\cap\mathcal{F}$$
$$G_r(\mathcal{C},\mathcal{W}\cap\mathcal{F})\subseteq\mathcal{W}\cap\mathcal{F}$$

here I decided to denote fibrations in $\mathcal{N}^{op}$ by $\mathcal{C}$ since they are cofibrations in $\mathcal{N}$

Proof. This boils down to $F_\lrcorner(i,j)\,\square\,p \Leftrightarrow i\,\square\,G_r(j,p)$ that must be checked and is tedious but completely elementary. $\square$

Definition. A *monoidal model category* is a category with a monoidal structure, i.e. $\otimes$ is a left biadjoint

$A\otimes B\to C$
$A\to \hom_r(B,C)$
$B\to \hom_\ell(A,C)$

Definition. A monoidal model category is a category with
- a closed monoidal structure, i.e. $\otimes$ is a left biadjoint with right adjoints $\hom_r$ and $\hom_\ell$

$A \otimes B \to C$ / $A \to \hom_r(B, C)$ / $B \to \hom_\ell(A, C)$

symmetric for us $\to$ $\hom_r = \hom_\ell = \{\ ,\ \}$

- a model structure

such that
- the unit of the monoidal structure S is either cofibrant or, more generally, the cofibrant replacement $S^c \xrightarrow{\sim} S$ tensors with all cofibrant objects to w.e.'s

$$S^c \otimes X \xrightarrow{\sim} S \otimes X \cong X$$
$$X \otimes S^c \xrightarrow{\sim} X \otimes S \cong X$$

this does not depend on the choice of the cofibrant replacement in view of

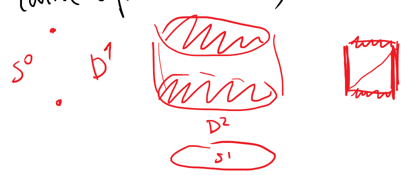
- the tensor product bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is left Quillen.

Examples. All $\underbrace{\mathrm{Cat}, \mathrm{Ch}, \mathrm{Top}}$, sSet are monoidal model categories.

cat of compactly gen. (weak) Hausdorff spaces

(with cofibrant unit)

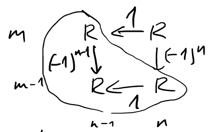
i_{n+m} v Top



$$\left.\begin{aligned} i_n &: S^{n-1} \to D^n \\ j_0 &: \cdot \to I \\ j_n &= j_0 \otimes_\lrcorner i_n \end{aligned}\right\}$$

$i_n \otimes_\lrcorner i_m$ is a cofibration

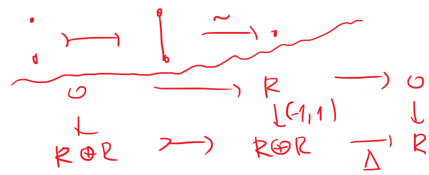
$\left.\begin{aligned} j_n \otimes_\lrcorner i_m \\ i_n \otimes_\lrcorner j_m \end{aligned}\right\}$ are trivial cofibrations



A functorial cylinder object for cofibrant objects: pick a cylinder object for the unit $S \in \mathcal{V}_c$: $S + S \rightarrowtail \mathrm{Cyl}\, S \xrightarrow{\sim} S$ and tensor it with $A \in \mathcal{V}_c$ to obtain

a w.e. between cofibrant objects

$$\begin{array}{ccccc} (S+S) \otimes A & \rightarrowtail & \mathrm{Cyl}\, S \otimes A & \xrightarrow{\sim} & S \otimes A \\ \| & & \|\,\mathrm{def} & & \| \\ A + A & \rightarrowtail & \mathrm{Cyl}\, A & \xrightarrow{\sim} & A \end{array}$$



Dually a path object is $\{\mathrm{Cyl}\, S, X\}$.

Remark. tensor-hom adjunction

$$\frac{A \otimes B \to C}{A \to \mathrm{Hom}(B, C)} \quad \text{sets of maps} \qquad \mathcal{V}(B, C) \in \mathrm{Set}$$

better: objects of maps $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$ $\leftarrow$ works since

$$\frac{X \to \mathrm{Hom}(A \otimes B, C)}{\dfrac{X \otimes A \otimes B \to C}{\dfrac{X \otimes A \to \mathrm{Hom}(B, C)}{X \to \mathrm{Hom}(A, \mathrm{Hom}(B, C))}}}$$

If $\mathcal{M}$ is enriched over $\mathcal{V}$, this can be generalized:

$$\mathcal{M}(K \otimes M, N) \cong \mathcal{M}(M, \{K, N\}) \qquad \{-,-\} : \mathcal{V}^{op} \times \mathcal{M} \to \mathcal{M}$$
$$\cong \qquad \not\cong$$
$$\mathcal{V}(K, \mathcal{M}(M, N)) \qquad \leftarrow \text{two different Hom's !}$$

$\mathcal{M}(-,-) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{V}$

Associativity: $\underset{\mathcal{V}}{\cap}\ \underset{\mathcal{M}}{\cap}\ \underset{\mathcal{M}}{\cap}$

$$\mathcal{M}(K \otimes (L \otimes M), N) \cong \mathcal{V}(K, \mathcal{M}(L \otimes M, N)) \cong \mathcal{V}(K, \mathcal{V}(L, \mathcal{M}(M, N)))$$
$$\cong \mathcal{V}(K \otimes L, \mathcal{M}(M, N)) \cong \mathcal{M}((K \otimes L) \otimes M, N)$$
$$\Rightarrow K \otimes (L \otimes M) \cong (K \otimes L) \otimes M$$

Unit $S \otimes M \cong M$

Definition. A $\mathcal{V}$-category $\mathcal{M}$ is said to be tensored if

$$\mathcal{V}(K, \mathcal{M}(M,N)) \cong \mathcal{M}(K\otimes M, N) \quad \text{naturally}$$

This gives a functor $\otimes : \mathcal{V}_0 \times \mathcal{M}_0 \to \mathcal{M}_0$ (in fact a $\mathcal{V}$-functor $\mathcal{V}\otimes\mathcal{M} \to \mathcal{M}$) that makes $\mathcal{M}_0$ into a "module" over $\mathcal{V}_0$. Dually $\mathcal{M}$ is cotensored if $\mathcal{V}(K, \mathcal{M}(M,N)) \cong \mathcal{M}(M, \{K,N\})$ naturally

This gives a functor $\{\,,\}: \mathcal{V}_0^{op} \times \mathcal{M}_0 \to \mathcal{M}_0$

Together with the hom-functor $\mathcal{M}(-,-): \mathcal{M}_0\times\mathcal{M}_0 \to \mathcal{V}_0$ these yield an adjunction of two variables.

If $A\otimes S^c \xrightarrow{\sim} A\otimes S = A$ for $A\in\mathcal{M}_c$ and this adjunction is Quillen, we say that $\mathcal{M}$ is a model $\mathcal{V}$-category.

Again $\mathrm{Cyl}\,S\otimes A$ is a cylinder object for $A\in\mathcal{M}_c$
$\{\mathrm{Cyl}\,S, X\}$ is a path object for $X\in\mathcal{M}_f$.

Left adjoint $\mathcal{V}$-functors preserve these cylinder objects (they are colimits),
right adjoint $\mathcal{V}$-functors preserve these path objects (they are limits).

Deriving bifunctors.

$F: \mathcal{M}\times\mathcal{N} \to \mathcal{P}$ left Quillen bifunctor

$\Rightarrow F: \mathcal{M}_c\times\mathcal{N}_c \to \mathcal{P}_c$ preserves weak equivalences:

$$\begin{matrix} 0 & 0 \\ \downarrow & \downarrow \\ M & N \end{matrix} \quad \rightsquigarrow \quad \begin{tikzcd} F(0,0) \arrow[r] \arrow[d] & F(M,0) \arrow[d] \arrow[dr] & \\ F(0,N) \arrow[r] \arrow[drr] & \cdot \arrow[dr] & \\ & & F(M,N) \end{tikzcd} \qquad F_\lrcorner(M,N): 0 \rightarrowtail F(M,N)$$

$$\begin{matrix} M' & 0 \\ i \downarrow\sim & \downarrow \\ M & N \end{matrix} \quad \rightsquigarrow \quad \begin{tikzcd} F(M',0) \arrow[r] \arrow[d] & F(M,0) \arrow[d] \arrow[dr] & \\ F(M',N) \arrow[r] \arrow[drr] & \cdot \arrow[dr, "\sim"] & \\ & & F(M,N) \end{tikzcd} \qquad F_\lrcorner(i,N): F(M',N) \overset{\sim}{\underset{F(i,N)}{\rightarrowtail}} F(M,N)$$

now apply Brown's lemma

Thus we get
$$\begin{tikzcd} \mathbb{L}F: Ho(\mathcal{M})\times Ho(\mathcal{N}) \arrow[r] & Ho(\mathcal{P}) \\ \mathcal{M}\times\mathcal{N} \arrow[u, "\gamma\times\gamma"] \arrow[r, "F(Q\times Q)"'] & \mathcal{P} \arrow[u, "\gamma"'] \end{tikzcd}$$
induced by

In particular, for a monoidal model category $\mathcal{V}$, we get

$$\otimes^{\mathbb{L}} : Ho(\mathcal{V})\times Ho(\mathcal{V}) \to Ho(\mathcal{V})$$

making the homotopy category into a monoidal closed category

$$\begin{matrix} (K\otimes^{\mathbb{L}} L)\otimes^{\mathbb{L}} M & \overset{?}{\cong} & K\otimes^{\mathbb{L}}(L\otimes^{\mathbb{L}} M) \\ \| & & \| \\ Q(QK\otimes QL)\otimes QM & & QK\otimes Q(QL\otimes QM) \end{matrix}$$

unit again S

the "weird" axiom : $X\in\mathcal{V}_c$

$$\begin{tikzcd}
Q(QK \otimes QL) \otimes QM \arrow[d, "q \otimes 1"', "\cong"] & QK \otimes Q(QL \otimes QM) \arrow[d, "1 \otimes q"', "\cong"] \\
(QK \otimes QL) \otimes QM \arrow[r, phantom, "\cong"] & QK \otimes (QL \otimes QM)
\end{tikzcd}$$

the "weird" axiom : $X \in \mathcal{V}_c$

$$\begin{tikzcd}
QS \otimes X \arrow[r] & S \otimes X \cong X \\
QS \otimes QX \arrow[u, "\cong"', "1 \otimes q"] \arrow[ur, "\cong", "q \otimes q"'] & \\
S \otimes^L X \arrow[u, equal] \arrow[uur, bend right, "\text{unit } \underline{\text{iso}}"'] &
\end{tikzcd}$$

More generally, if $\mathcal{M}$ is a model $\mathcal{V}$-category, then $Ho(\mathcal{M})$ is

- a category enriched in $Ho(\mathcal{V})$
- tensored and cotensored

$\Big\}$ need to derive the Hom's $\mathcal{M}^{\mathbb{R}}(-,-)$, $\{-,-\}^{\mathbb{R}}$

Reedy model categories, framings

$\mathcal{M}(M,N)$

Motivation. Any model category $\mathcal{M}$ is in some weak sense enriched over sSet and, as a result, $Ho(\mathcal{M})$ will be enriched over $Ho(sSet)$.

Start with an honest sSet-model category = simplicial model category

- since $K \in sSet$ is a colimit $K = \operatorname{colim}_{(n, \Delta^n \to K)} \Delta^n$

  we have $K \otimes M = \operatorname{colim}(\Delta^n \otimes M)$ and it is enough to give $\Delta^n \otimes M$

- clearly $\Delta^0 \otimes M \cong M$, since Δ^0 is the monoidal unit
- what is the essential property of $\Delta^1 \otimes M$?

  it is a cylinder ... at least for $M \in \mathcal{M}_c$:

$$\Delta^0 + \Delta^0 \overset{[d^0, d^1]}{\rightarrowtail} \Delta^1 \xrightarrow[\sim]{s^0} \Delta^0 \quad \rightsquigarrow \quad M + M \rightarrowtail \Delta^1 \otimes M \xrightarrow{\sim} M$$

- what about $\Delta^2 \otimes M$?

$$\begin{tikzcd}
 & & \Delta^0 \arrow[dl, "d^0"'] \arrow[dr, "d^0"] & & \\
 & \Delta^1 \arrow[r, "d^1"] & \Delta^2 & \Delta^1 \arrow[l, "d^0"'] & \\
\Delta^0 \arrow[ur, "d^1"] \arrow[rr, "d^1"'] & & \Delta^1 \arrow[u, "d^2"] & & \Delta^0 \arrow[ll, "d^0"] \arrow[ul, "d^1"']
\end{tikzcd}$$

, more compactly $\partial\Delta^2 \to \Delta^2$, $\partial\Delta^2 = L_2\Delta^\bullet$

$\rightsquigarrow L_2\Delta^\bullet \otimes M \rightarrowtail \Delta^2 \otimes M \xrightarrow{\sim} M_2\Delta^\bullet \otimes M$

$$\begin{tikzcd}
 & \Delta^2 \arrow[dl, "s^0"'] \arrow[dr, "s^1"] & \\
\Delta^1 \arrow[dr, "s^0"'] & & \Delta^1 \arrow[dl, "s^0"] \\
 & \Delta^0 &
\end{tikzcd}$$

, more compactly $\Delta^2 \xrightarrow{\sim} M_2\Delta^\bullet$ $\quad \Delta^\bullet : \Delta \longrightarrow sSet$ a cosimplicial object in sSet

- need some calculus of such diagrams $\Delta \to sSet$ $\quad \rightsquigarrow \Delta^\bullet \otimes M : \Delta \to \mathcal{M}$ a cosimplicial object in $\mathcal{M}$

  $\to$ Reedy categories, Reedy model structures $\Delta \to \mathcal{M}$

A Reedy category has two kinds of maps – direct and inverse (like d^i and s^i in Δ)

Definition. A direct category is a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A} \to \lambda$ (ordinal) that satisfies $f : A \to B \rightsquigarrow f = 1 \Leftrightarrow \deg A = \deg B$

Now we denote by $\mathcal{A}_a \in [\mathcal{A}, Set]$ the representable $\mathcal{A}(a, -)$

Dually $\mathcal{A}^a \in [\mathcal{A}^{op}, Set]$ denotes $\mathcal{A}(-, a)$ and we define

- $\partial\mathcal{A}^a \subseteq \mathcal{A}^a$ a subfunctor obtained by removing 1
- $i_a : \partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a$ the inclusion

We say that $f : X \to Y$ is a (trivial) cofibration if $i_a \otimes_{\mathcal{A}} f$ is a (trivial) cofibration