Weighted colimits

tensor adjunction (or any other) $\mathrm{Hom}_R(K, \mathrm{Hom}(M,N)) \cong \mathrm{Hom}(K \otimes_R M, N)$
$[R^{op}, Ab]$ Ab Ab

$$\mathcal{V}(K, \mathcal{M}(M,N)) \cong \mathcal{M}(K \otimes M, N)$$

, e.g. abelian groups and their $\otimes$

how about $\otimes_R$?

replace by diagrams

$\bullet_R \longmapsto M$, $\xrightarrow{\text{ring homo}} \mathrm{End}(M)$

R-module = $R \longrightarrow Ab$ left

weight $W : \mathcal{A}^{op} \to \mathcal{V}$ $\quad D : \mathcal{A} \to \mathcal{M}$ diagram

$$[\mathcal{A}^{op}, \mathcal{V}](W, \underbrace{\mathcal{M}(D, N)}) \cong \mathcal{M}(W \otimes_{\mathcal{A}} D, N)$$

$\mathcal{A}^{op} \xrightarrow{D^{op}} \mathcal{M}^{op} \xrightarrow{\mathcal{M}(-,N)} \mathcal{V}$

$R = \bullet_R$ $\quad R^{op} \longrightarrow Ab$ right

Hom_R = hom in $[R^{op}, Ab]$?

Definition The weighted colimit $W \otimes_{\mathcal{A}} D$ ... the colimit of D weighted by W is the object representing $[\mathcal{A}^{op}, \mathcal{V}](W, \mathcal{M}(D, -))$ as above.

Examples • The tensor product of R-modules

• the tensors ... take $\mathcal{A} = \bullet_S$ were $S \in \mathcal{V}$ is the unit $\quad W_* = K$
$W \otimes_{\mathcal{A}} M = K \otimes M$

• The geometric realization $\quad$ tensor over Set: $K \cdot P = \coprod_K P$

$$\mathrm{Set}(K, \mathrm{Top}(P, X)) \cong \mathrm{Top}(K \cdot P, X)$$

make Δ-indexed

$$\mathrm{sSet}(K_\bullet, \underbrace{\mathrm{Top}(|\Delta^\bullet|, X)}_{SX}) \cong \mathrm{Top}(\underbrace{K_\bullet \cdot_\Delta |\Delta^\bullet|}_{|K|}, X)$$

$\Delta^{op} \xrightarrow{K_\bullet} \mathrm{Set}$ weight

$\Delta \xrightarrow[\text{"Yoneda"}]{|\Delta^\bullet|} \mathrm{Top}$ diagram

$n \mapsto |\Delta^n|$

• The ordinary colimits: $\mathcal{V} = \mathrm{Set}$ $\quad W = \Delta *$

$$\underbrace{[\mathcal{A}^{op}, \mathrm{Set}](\Delta *, \mathcal{M}(D, N))}_{\mathrm{cone}(D,N)} \cong \mathcal{M}(\Delta * \otimes_{\mathcal{A}} D, N)$$

$\Delta * (A) : \quad DA \to N$, $\quad DB \to N$

The general weighted colimits can be translated to ordinary colimits

$$W \otimes_{\mathcal{A}} D = \operatorname*{colim}_{\mathrm{El}\,W} DP \qquad \mathrm{El}\,W \xrightarrow{P} \mathcal{A} \xrightarrow{D} \mathcal{M}$$

"take DA multiple times - once for each element of WA" $\quad \bullet\ A \quad W_\bullet = K$

Another point of view:

$\Delta_n = \Delta(n, -)$
$\Delta^n = \Delta(-, n)$
$\#K \times$ $\{$ El W

• $[\mathcal{A}^{op}, \mathcal{V}](W, \mathcal{M}(D, N)) \cong \mathcal{M}(W \otimes_{\mathcal{A}} D, N)$

take $W = \mathcal{A}(-, A) = \mathcal{A}^A$ representable functor

$\Rightarrow$ get on the left $\mathcal{M}(D, N)(A) = \mathcal{M}(DA, N)$ by Yoneda

$\mathcal{M}(D-, N)(A) =$

$\Rightarrow \mathcal{A}^A \otimes_{\mathcal{A}} D = DA$

• clearly $W \otimes_{\mathcal{A}} D$ preserves colimits (ordinary or weighted) in the W-variable $\Rightarrow$ for $\mathcal{V} = \mathrm{Set}$ any weight = presheaf is a colimit of representables $\Rightarrow$ weighted colimits can be replaced by colimits (over the cat of elements)

$\longrightarrow \mathcal{V} = Ab$, $\quad \mathcal{A} = a \underset{\mathbb{Z}}{\rightleftarrows} b$ ($\mathbb{Z}\cdot 1_a$, $\mathbb{Z}\cdot 1_b$) $\Rightarrow [\mathcal{A}, \mathcal{M}]$ = arrows of $\mathcal{M}$

$\longrightarrow \mathcal{V} = Ab$, $\mathcal{A} = \begin{tikzcd} a \arrow[r, "\mathbb{Z}"] \arrow[loop above, "\mathbb{Z}\cdot 1"] & b \arrow[loop above, "\mathbb{Z}\cdot 1_b"] \end{tikzcd}$ $\Rightarrow [\mathcal{A}, \mathcal{M}]$ = arrows of $\mathcal{M}$

$\mathcal{A} \to [\mathcal{A}^{op}, Ab]$

$a \xrightarrow{f} b$, $\mathcal{A}^a \xrightarrow{in} \mathcal{A}^b$ colimit

$\mathcal{A}^a \xrightarrow{in} \mathcal{A}^b \to \mathcal{A}^b/\mathcal{A}^a$

$$\mathcal{A}^a = \mathbb{Z} \leftarrow 0$$
$$\mathcal{A}^b = \mathbb{Z} \xleftarrow{1} \mathbb{Z}$$
$$\mathcal{A}^b/\mathcal{A}^a = 0 \leftarrow \mathbb{Z}$$

$\rightsquigarrow$

$$\mathcal{A}^a \otimes_{\mathcal{A}} D = Da$$
$$\mathcal{A}^b \otimes_{\mathcal{A}} D = Db$$
$$\mathcal{A}^b/\mathcal{A}^a \otimes_{\mathcal{A}} D = \mathrm{coker}(Da \to Db)$$

coeq's: $a \underset{g}{\overset{f}{\rightrightarrows}} b$ $\quad \mathcal{A}^a = \{1\} \leftleftarrows \emptyset$, $\mathcal{A}^b = \{f, g\} \leftleftarrows \{1\}$, coeq $= * \leftleftarrows *$

pushouts: $\begin{tikzcd} a \arrow[r] \arrow[d] & b \\ c \end{tikzcd}$ $\Delta * = \mathrm{colim}(\mathcal{A}^a \rightrightarrows \mathcal{A}^b + \mathcal{A}^c)$

$\begin{tikzcd} * \arrow[r] & * \\ * \arrow[u] \end{tikzcd} = \mathrm{colim}\left(\begin{tikzcd} * & \emptyset \arrow[l] \\ \emptyset \arrow[u] \end{tikzcd} \rightrightarrows \left(\begin{tikzcd} * & * \arrow[l] \\ \emptyset \arrow[u] \end{tikzcd} + \begin{tikzcd} * & \emptyset \arrow[l] \\ * \arrow[u] \end{tikzcd}\right)\right)$

- The conical colimits "=" the ordinary colimits in the underlying category $\mathcal{M}_0$: $\cdot S : Set \rightleftarrows \mathcal{V}_0 : \mathcal{V}_0(S, -)$ tensor-hom adjunction

$\mathbb{Z} : Set \rightleftarrows Ab : Ab_0(\mathbb{Z}, -)$
free ab. gp

$\mathcal{A}$ an ordinary category

$\to \mathcal{A}\cdot S$ a $\mathcal{V}$-category that admits a canonical weight $\Delta * \cdot S = \Delta S$

$$[\mathcal{A}^{op}, Set](\Delta *, \mathcal{M}_0(D, N)) \cong \mathcal{M}_0(\Delta * \cdot_{\mathcal{A}} D, N)$$
$$\cong$$
$$[\mathcal{A}^{op}, \mathcal{V}_0](\Delta S, \mathcal{M}(D, N))$$
$$\cong$$
$$[(\mathcal{A}\cdot S)^{op}, \mathcal{V}]_0(\Delta S, \mathcal{M}(D, N))$$

this is almost saying that $\Delta * \cdot_{\mathcal{A}} D = \Delta S \otimes_{\mathcal{A}S} D$ up to the index 0

Lemma. If $\mathcal{M}$ has cotensors, this works in the enriched sense

Proof. We need $[(\mathcal{A}\cdot S)^{op}, \mathcal{V}](\Delta S, \mathcal{M}(D, N)) \cong \mathcal{M}(\Delta * \cdot_{\mathcal{A}} D, N)$

$\overset{\text{Yoneda}}{\Longleftrightarrow}$ $\mathcal{V}_0(\underset{S \otimes K}{K}, [(\mathcal{A}\cdot S)^{op}, \mathcal{V}](\Delta S, \mathcal{M}(D, N))) \cong \mathcal{V}_0(K, \mathcal{M}(\Delta * \cdot_{\mathcal{A}} D, N))$

$$\mathcal{V}_0(S, [(\mathcal{A}\cdot S)^{op}, \mathcal{V}](\Delta S, \mathcal{M}(D, N^K))) \qquad \mathcal{V}_0(S, \mathcal{M}(\Delta * \cdot_{\mathcal{A}} D, N^K))$$

$$[(\mathcal{A}\cdot S)^{op}, \mathcal{V}]_0(\Delta S, \mathcal{M}(D, N^K)) \qquad \mathcal{M}_0(\Delta * \cdot_{\mathcal{A}} D, N^K) \qquad \square$$

Summary. ordinary colimits $\overset{\text{often}}{=}$ conical colimits $\subseteq$ weighted colimits $\supseteq$ tensors

Construction. $W \otimes_{\mathcal{A}} D$ can be constructed as the coequalizer of

$$\sum_{B,C} WC \otimes \mathcal{A}(B, C) \otimes DB \underset{1 \otimes act}{\overset{act \otimes 1}{\rightrightarrows}} \sum_{A} WA \otimes DA$$

like $M \otimes_R N$

the coend

$[\mathcal{A}^{op}, \mathcal{V}](F, G) = \int_{\mathcal{A}} \mathcal{V}(F-, G-)$, $\mathcal{A}^{op} \times \mathcal{A} \to \mathcal{V}$

Explanation: $[\mathcal{A}^{op}, \mathcal{V}](W, \mathcal{M}(D, N))$ is the equalizer of

$$\prod_A \mathcal{V}(WA, \mathcal{M}(DA, N)) \rightrightarrows \prod_{B,C} \{\mathcal{A}(B, C), \mathcal{V}(WC, \mathcal{M}(DB, N))\}$$

$\begin{tikzcd} WB \arrow[r] & \ldots B \\ WC \arrow[u] \arrow[ur] \arrow[r] & \ldots C \arrow[u] \end{tikzcd}$

$$\cong \qquad\qquad\qquad\qquad \cong$$

$$\prod \mathcal{M}(WA \otimes DA, N) \qquad \prod \mathcal{M}(WC \otimes \mathcal{A}(B, C) \otimes DB, N)$$

$$\prod_A \mathcal{M}(WA\otimes DA, N) \qquad \prod_{B,C} \mathcal{M}(WC\otimes \mathcal{A}(B,C)\otimes DB, N)$$

obtained from the paraller pair from the construction upon applying $\mathcal{M}(-,N)$ that turns colimits into limits. $\square$

Example. $\mathcal{M}$ a simplicial category; then the geometric realization is a functor $|-|: s\mathcal{M}\to\mathcal{M}$ given by

$|X| = \Delta^\bullet \otimes_\Delta X$ ($\Delta^\bullet: \Delta \to sSet$ weight; $X: \Delta^{op}\to\mathcal{M}$ diag) the colimit weighted by $\Delta^\bullet: \Delta \to sSet$ " Yoneda embedding

We will see that $\Delta^\bullet$ is Reedy cofibrant.

The $-\otimes_\Delta -: [\Delta, sSet]\times[\Delta^{op},\mathcal{M}]\to\mathcal{M}$ is left Quillen w.r.t. Reedy structures $\Rightarrow$ $|-|$ homotopy invariant on Reedy cofibrant diag's

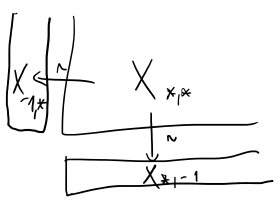
E.g. for $\mathcal{M} = sSet \Rightarrow |X| = \operatorname{diag} X$

$$|X| = \int^n \Delta^n \otimes \underset{\sim}{X_n}$$
$$\int^k \Delta^k \otimes X_{nk}$$
$$= \int^{n,k} (\Delta^n\times\Delta^k)\otimes X_{nk}$$
$$= \int^{n,k} \Delta\times\Delta(\operatorname{diag}-,(n,k))\otimes X_{nk}$$
$$= X\circ \operatorname{diag}$$

Any X is Reedy cofibrant

If all augmentations $X_n \xrightarrow{\sim} X_{-1}$ are w.e.

$\Rightarrow X \xrightarrow{\sim} cst\, X_{-1} \Rightarrow |X| \xrightarrow{\sim} |cst\, X_{-1}| = X_{-1}$



"balancing"

$$\Delta^{op} \xrightarrow{\text{diag}} \Delta^{op}\times\Delta^{op} \xrightarrow{X} Set$$
$$X\circ\text{diag}$$

Further interesting topics. Restriction and extension of scalars

$$F_!: [\mathcal{A},\mathcal{M}] \underset{\perp}{\rightleftarrows} [\mathcal{B},\mathcal{M}]: F^* \qquad F:\mathcal{A}\to\mathcal{B}$$

Lan_F $\qquad \mathcal{B}(-,F)\otimes_{\mathcal{A}} - \qquad\qquad \mathcal{B}(F,-)\otimes_{\mathcal{B}} -$

${}_S S_R \otimes_R - \qquad\qquad {}_R S_S \otimes_S -$

Reminder on weighted colimits

$$X \times_G Y \qquad M \otimes_R N \qquad W \otimes_{\mathcal A} D \qquad \Delta^\bullet \times_\Delta X$$

$$X\times G\times Y \rightrightarrows X\times Y \qquad M\otimes R\otimes N \rightrightarrows M\otimes N \qquad \sum_{b,c} Wc\otimes \mathcal A(b,c)\otimes Db \rightrightarrows \sum_a Wa\otimes Da \qquad ---- \rightrightarrows \sum \Delta^n\times X_n$$

$X: G^{op}\to Set$, $Y: G\to Set$

$M: R^{op}\to Ab$, $N: R\to Ab$

$W:\mathcal A^{op}\to\mathcal V$, $D:\mathcal A\to\mathcal M$

$\Delta^\bullet:\Delta\to sSet$, $X:\Delta^{op}\to Top$ ....

$$\sum_{b,c} Wc\otimes\mathcal A(b,c)\otimes Db \rightrightarrows \sum_a Wa\otimes Da \xrightarrow[\text{coeq}]{} W\otimes_{\mathcal A} D \qquad / \ \mathcal M(-,N)$$

$$\begin{tikzcd}
\prod_{b,c}\{\mathcal A(b,c),\mathcal M(Wc\otimes Db,N)\} \arrow[d, equal] & \prod_a \mathcal M(Wa\otimes Da,N) \arrow[l, shift left] \arrow[l, shift right] \arrow[d, equal] & \mathcal M(W\otimes_{\mathcal A}D,N) \arrow[l] \\
\prod_{b,c}\{\mathcal A(b,c),\mathcal V(Wc,\mathcal M(Db,N))\} & \prod_a \mathcal V(Wa,\mathcal M(Da,N)) \arrow[l, shift left] \arrow[l, shift right] & {[\mathcal A^{op},\mathcal V](W,\mathcal M(D,N))} \arrow[l, "\text{end}"']
\end{tikzcd}$$

Example.

$\mathcal A(-,A)\otimes_{\mathcal A} D = DA$ dually $\{\mathcal A(A,-),D\}_{\mathcal A} = DA$

$\mathcal A(-,-)\otimes_{\mathcal A} D = D \leftarrow R\otimes_R M = M \qquad Hom_R(R,M)=M$

$$\left.\begin{array}{l} W:\mathcal A^{op}\times\mathcal B\to\mathcal V \\ D:\mathcal A\to\mathcal M \end{array}\right\} \quad W\otimes_{\mathcal A}D:\mathcal B\to\mathcal M$$

Restriction of scalars: $F:\mathcal B\to\mathcal A$

$\mathcal A(-,F-):\mathcal A^{op}\times\mathcal B\to\mathcal V$

$\mathcal A(-,F-)\otimes_{\mathcal A}D = DF$ dually $\{\mathcal A(F-,-),D\}_{\mathcal A} = DF = F^*D$

Adjunction:

$$[\mathcal B,\mathcal V]\big(C,\overbrace{\{\mathcal A(F-,-),D\}_{\mathcal A}}^{F^*D}\big) = [\mathcal A,\mathcal V]\big(\underbrace{\mathcal A(F-,-)\otimes_{\mathcal B} C}_{F_!C}, D\big)$$

$F_!C$ extension of scalars (like $S\otimes_R L$)

$$[\mathcal B,\mathcal V](C,F^*D) = [\mathcal A,\mathcal V](F_!C,D)$$

$\operatorname{Lan}_F C$ left Kan extension

e.g. $F:\mathcal B\hookrightarrow\mathcal A$ full embedding

$$F_! C(B) = \mathcal{A}(F-, B) \otimes_{\mathcal{B}} C$$

$$= \mathcal{B}(-, B) \otimes_{\mathcal{B}} C$$

$$= C_B \quad \text{real extension}$$

Reedy model categories, framings

$\mathcal{M}(M,N)$

Motivation. Any model category $\mathcal{M}$ is in some weak sense enriched over sSet and, as a result, $Ho(\mathcal{M})$ will be enriched over $Ho(sSet)$.

Start with an honest sSet-model category = simplicial model category

– since $K \in sSet$ is a colimit $K = \operatorname{colim}_{(n, \Delta^n \to K)} \Delta^n$

we have $K \otimes M = \operatorname{colim}(\Delta^n \otimes M)$ and it is enough to give $\Delta^n \otimes M$

– clearly $\Delta^0 \otimes M \cong M$, since Δ^0 is the monoidal unit

– what is the essential property of $\Delta^1 \otimes M$?

it is a cylinder ! ... at least for $M \in \mathcal{M}_c$:

$$\Delta^0 + \Delta^0 \xrightarrow{[d^0, d^1]} \Delta^1 \xrightarrow[\sim]{s^0} \Delta^0 \quad \rightsquigarrow \quad M + M \rightarrowtail \Delta^1 \otimes M \xrightarrow{\sim} M$$

– what about $\Delta^2 \otimes M$?

$$\begin{tikzcd}
 & & \Delta^0 \arrow[dl] \arrow[dr, "d^0"] & & \\
 & \Delta^1 \arrow[r, "d^1"] & \Delta^2 & \Delta^1 \arrow[l, "d^0"'] & \\
\Delta^0 \arrow[ur] \arrow[rr, "d^1"'] & & \Delta^1 \arrow[u, "d^2"] & & \Delta^0 \arrow[ll, "d^0"] \arrow[ul, "d^1"']
\end{tikzcd}$$

, more compactly $\partial\Delta^2 \to \Delta^2$, $\partial\Delta^2 = L_2\Delta^\bullet$

$\rightsquigarrow L_2\Delta^\bullet \otimes M \rightarrowtail \Delta^2 \otimes M \xrightarrow{\sim} M_2\Delta^\bullet \otimes M$

$$\begin{tikzcd}
 & \Delta^2 \arrow[dl, "s^0"'] \arrow[dr, "s^1"] & \\
\Delta^1 \arrow[dr, "s^0"'] & & \Delta^1 \arrow[dl, "s^0"] \\
 & \Delta^0 &
\end{tikzcd}$$

, more compactly $\Delta^2 \xrightarrow{\sim} M_2\Delta^\bullet$

$\Delta^\bullet : \Delta \longrightarrow sSet$ a cosimplicial object in sSet

– need some calculus of such diagrams $\Delta \to sSet$ $\rightsquigarrow \Delta^\bullet \otimes M : \Delta \to \mathcal{M}$ a cosimplicial object in $\mathcal{M}$

$\to$ Reedy categories, Reedy model structures $\Delta \to \mathcal{M}$

A Reedy category has two kinds of maps – direct and inverse (like d^i and s^i in Δ)

Definition A *direct category* is a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A} \to \lambda$ (ordinal) that satisfies $f : A \to B \rightsquigarrow f = 1 \Leftrightarrow \deg A = \deg B$

An *inverse category* is a dual notion, i.e. a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A}^{op} \to \lambda$ satisfying $f = 1 \Leftrightarrow \deg A = \deg B$.

A *Reedy category* is a category $\mathcal{A}$ together with two subcategories $\mathcal{A}_+, \mathcal{A}_-$ and a function $\deg : \operatorname{ob} \mathcal{A} \to \lambda$ that

- makes $\mathcal{A}_+$ into a direct category
- makes $\mathcal{A}_-$ into an inverse category
- any morphism has a unique decomposition:

$$\sum_{A \in \mathcal{A}} \mathcal{A}_+(A, C) \times \mathcal{A}_-(B, A) \xrightarrow[\cong]{\circ} \mathcal{A}(B, C)$$

as a composition of an inverse and a direct morphism

The Yoneda lemma gives $X \cong \mathcal{A}(-,-) \cdot_{\mathcal{A}} X$ and we will describe a way of building X by decomposing $\mathcal{A}(-,-)$. The axioms actually give

$$\mathcal{A}(-,-) = \sum_{A \in \mathcal{A}} \mathcal{A}^+(A,-) \times \mathcal{A}_-(-,A) = \sum_{A \in \mathcal{A}} \mathcal{A}^+_A \times \mathcal{A}^A_-$$

(but only of functors $\mathcal{A}_-^{op} \times \mathcal{A}^+ \to \mathrm{Set}$?)

but in order to understand $\mathcal{A}(-,-)$ we factor it into

$\mathrm{sk}_n \mathcal{A}(-,-)$ ... maps that factor through some A of degree $\leq n$

and clearly we have

$$\mathcal{A}(-,-) = \operatorname{colim}_{n<\lambda} \mathrm{sk}_n \mathcal{A}(-,-)$$

so that it remains to study the difference between $\mathrm{sk}_n \mathcal{A}$ and $\mathrm{sk}_{n-1} \mathcal{A}$ or, better for n limit, $\mathrm{sk}_{<n} \mathcal{A} = \operatorname{colim}_{i<n} \mathrm{sk}_i \mathcal{A}$.

Quite clearly, we have a pushout square

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{sk}_{<n} \mathcal{A} \arrow[d] \\
\sum_{A \in \mathcal{A}_n} \mathcal{A}^+_A \times \mathcal{A}^A_- \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{sk}_n \mathcal{A}
\end{tikzcd}$$

with A ranging over all objects of degree n

However, it will be crucial to express this in terms of representable functors on $\mathcal{A}$, rather than on $\mathcal{A}^+, \mathcal{A}_-$.

Examples

- any ordinal is a direct category
- $\bullet \overset{+}{\rightrightarrows} \bullet$ is a direct category
- $\bullet \xleftarrow{-} \bullet \xrightarrow{+} \bullet$ is a Reedy category
- Δ^+ is a direct category (non-empty finite ordinals + monos)
- Δ_- is an inverse category (non-empty finite ordinals + epis)
- Δ is a Reedy category

Notation

We denote

- $\mathcal{A}_A = \mathcal{A}(A,-) \in [\mathcal{A}, \mathrm{Set}]$ the covariant representable
- $\mathcal{A}^A = \mathcal{A}(-,A) \in [\mathcal{A}^{op}, \mathrm{Set}]$ the contravariant representable ... think Δ^n

We further define two subfunctors

- $i_A : \partial \mathcal{A}_A \subseteq \mathcal{A}_A$ of maps that factor through an object of lower degree.
- $i^A : \partial \mathcal{A}^A \subseteq \mathcal{A}^A$ of maps that factor through an object of lower degree.

In the decomposition

$$\mathcal{A}_A = \mathcal{A}(A,-) = \sum \mathcal{A}^+(B,-) \times \mathcal{A}_-(A,B)$$
$$\partial \mathcal{A}_A = \sum \mathcal{A}^+(B,-) \times \partial \mathcal{A}_-(A,B)$$

only 1 is excluded (coproduct)

This means that there is a pushout

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \partial \mathcal{A}_A \arrow[d, "i_A"] \\
\mathcal{A}^+_A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{A}_A
\end{tikzcd}$$

Dually, we get

$$\mathcal{A}^A = \mathcal{A}(-,A) = \sum \mathcal{A}^+(B,A) \times \mathcal{A}_-(-,B)$$
$$\partial \mathcal{A}^A = \sum \partial \mathcal{A}^+(B,A) \times \mathcal{A}_-(-,B) = \sum \underbrace{\partial (\mathcal{A}^+)^A \times_{\mathcal{A}^+} \mathcal{A}^+(B,-)}_{\partial \mathcal{A}^+(B,A) \text{ by Yoneda}} \times \mathcal{A}_-(-,B)$$

$$A^A = A(-,A) = \sum A^+(B,A) \times A_-(-,B)$$
$$\partial A^A = \sum \partial A^+(B,A) \times A_-(-,B) \quad \text{(coproduct)}$$

$$= \sum \partial(A^+)^A \times_{A^+} A^+(B,-) \times A_-(-,B)$$
$$= \partial(A^+)^A \times_{A^+} \underbrace{\sum A^+(B,-) \times A_-(-,B)}_{A(-,-)}$$
$$= \partial(A^+)^A \cdot_{A^+} A(-,-)$$
$$= \text{extension of } \partial(A^+)^A$$
$$\Rightarrow \partial A^A \cdot_A X = \left(\partial(A^+)^A \times_{A^+} A(-,-)\right) \cdot_A X = \partial(A^+)^A \cdot_{A^+} X$$

and a pushout

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \partial A^A \arrow[d, "i^A"] \\ A^A_- \arrow[r] & A^A \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Putting these together yields that

$$\begin{tikzcd} & 0 \times 0 \arrow[dl] \arrow[dr] & \\ 0 \times A^A_- \arrow[dr] \arrow[ddr] & & A^+_A \times 0 \arrow[dl] \arrow[ddl] \\ & \bullet \arrow[d] & \\ & A^+_A \times A^A_- & \end{tikzcd}$$

zero

each square pushout $\Longrightarrow$ gives us a pushout

$$\begin{tikzcd} & \partial A_A \times \partial A^A \arrow[dl] \arrow[dr] & \\ \partial A_A \times A^A \arrow[dr] \arrow[ddr] & & A_A \times \partial A^A \arrow[dl] \arrow[ddl] \\ & \bullet \arrow[d, "i_A \square i^A"] & \\ & A_A \times A^A & \end{tikzcd}$$

Now we get a diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \sum_A \partial A_A \times A^A +_{\partial A_A \cdot \partial A^A} A_A \times \partial A^A \arrow[r] \arrow[d, "\sum_A i_A \square i^A"] & \mathrm{sk}_{<n} A(-,-) \arrow[d] \\ \sum_A A^+_A \times A^A_- \arrow[r] & \sum_A A_A \times A^A \arrow[r] \arrow[ul, phantom, "\ulcorner"] & \mathrm{sk}_n A(-,-) \end{tikzcd}$$

sums range over $A \in A$ with $\deg A = n$

in which the outer square is a pushout $\Longrightarrow$ so is the one on the right but now in $[A^{op} \times A, \mathrm{Set}]$!

Upon applying $- \cdot_A X$, we denote:

$$\begin{tikzcd} \partial A^A \cdot_A X \arrow[r, "i^A \cdot_A X"] \arrow[d, equal, "\mathrm{def}"] & A^A \cdot_A X \arrow[d, equal] \\ L_A X \arrow[r, "\partial_A X"] & X_A \end{tikzcd}$$

Example. $A = \Delta^{op}$, $M = \mathrm{Set}$
$\Rightarrow L_n X \subseteq X_n$ the subset of deg. simplices (needs a bit of work, see above).

$0 = \mathrm{sk}_{-1} X$

$X = \mathrm{colim}\, \mathrm{sk}_n X$ and

Theorem. For any $X \in [A, M]$ we get

$$\begin{tikzcd} \sum_A \partial A_A \cdot X_A +_{\partial A_A \cdot L_A X} A_A \cdot L_A X \arrow[r] \arrow[d, "i_A \square \partial_A X"] & \mathrm{sk}_{<n} X \arrow[d] \\ \sum A_A \cdot X_A \arrow[r] & \mathrm{sk}_n X \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Important special case.

When A is direct, we have $\partial A_A = 0$ and, consequently,

$$\begin{tikzcd} \sum_A A_A \cdot L_A X \arrow[r] \arrow[d] & \mathrm{sk}_{<n} X \arrow[d] \\ \sum_A A_A \cdot X_A \arrow[r] & \mathrm{sk}_n X \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

so that: $\forall A \in A$: $\partial_A X : L_A X \to X_A$ cofibration

$\Longrightarrow X$ is cofibrant in the projective model structure

More generally, for $f: X \to Y$ we denote $\partial_A f = i^A \square_A f$, i.e. the pushout corner map in

$$\begin{tikzcd} L_{\mathcal{A}}X \arrow[r] \arrow[d, "\partial_A X"'] & L_{\mathcal{A}}Y \arrow[d] \arrow[ddr, bend left, "\partial_A Y"] & \\ X_A \arrow[r] \arrow[drr] & \cdot \arrow[dr] & \\ & & Y_A \end{tikzcd} \qquad \partial_A f = \begin{matrix} \partial \mathcal{A}^A \\ \downarrow i^A \\ \mathcal{A}^A \end{matrix} \ \hat{\cdot}_{\mathcal{A}} \ \begin{matrix} X \\ \downarrow f \\ Y \end{matrix}$$

Theorem. For any map $f: X \to Y$ of $[\mathcal{A}, \mathcal{M}]$ we get $X = sk_{-1}^X Y$ and $Y = \operatorname{colim}_{n<\lambda} sk_n^X Y$ and

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\Sigma i_A \hat{\cdot}_{\mathcal{A}} \partial_A f"'] & sk_{<n}^X Y \arrow[d] \\ \cdot \arrow[r] & sk_n^X Y \end{tikzcd} \qquad \begin{matrix} 0 \\ \downarrow \\ sk_n \mathcal{A}(-,-) \\ \downarrow \\ \mathcal{A}(-,-) \end{matrix} \ \hat{\cdot}_{\mathcal{A}} \ \begin{matrix} X \\ \downarrow f \\ Y \end{matrix} \ = \ \begin{matrix} X \\ \downarrow \\ sk_n^X Y = sk_n Y +_{sk_n X} X \\ \downarrow \\ Y \end{matrix}$$

Theorem. For a map $f: X \to Y$ of $[\mathcal{A}, \mathcal{M}]$ TFAE

- $\forall A$: $\partial_A f$ is a (trivial) cofibration
- $f \in \operatorname{cell}\{ i_A \hat{\cdot} i \mid A \in \mathcal{A},\ i \in \mathcal{M}$ a (trivial) cofibration$\}$ $\leftarrow$ $\begin{matrix} \partial \mathcal{A}_A \cdot L +_{\partial \mathcal{A}_A \cdot K} \mathcal{A}_A \cdot K \\ \downarrow \\ \mathcal{A}_A \cdot L \end{matrix}$

These maps are called Reedy (trivial) cofibrations.

Proof. The implication "$\Downarrow$" is the previous theorem.
For the implication "$\Uparrow$" observe that ∂_A commutes with all cellular constructions so that it is enough to study $\partial_A (i_B \hat{\cdot} i)$:

$$\begin{matrix} \partial \mathcal{A}^A \\ \downarrow i^A \\ \mathcal{A}^A \end{matrix} \ \hat{\cdot}_{\mathcal{A}} \left(\begin{matrix} \partial \mathcal{A}_B \\ \downarrow i_B \\ \mathcal{A}_B \end{matrix} \ \hat{\cdot} \ \begin{matrix} K \\ \downarrow i \\ L \end{matrix} \right) = \underbrace{\left(\begin{matrix} \partial \mathcal{A}^A \\ \downarrow i^A \\ \mathcal{A}^A \end{matrix} \ \hat{\times}_{\mathcal{A}} \ \begin{matrix} \partial \mathcal{A}_B \\ \downarrow i_B \\ \mathcal{A}_B \end{matrix} \right)}_{\substack{\text{either iso for } A \neq B \\ \mathcal{A}_{\neq 1}(B,A) \hookrightarrow \mathcal{A}(B,A) \text{ for } A = B \\ \Downarrow \\ \text{pushout of } 0 \to 1}} \hat{\cdot} \ \begin{matrix} K \\ \downarrow i \\ L \end{matrix}$$

$\underbrace{\qquad\qquad}_{\text{pushout of } i}$ $\square$

Dually, we denote

$$\begin{matrix} M_A X = \{\partial \mathcal{A}_A, X\}_{\mathcal{A}} \\ \delta_A X \uparrow \qquad \qquad \uparrow \{i_A, X\}_{\mathcal{A}} \\ X_A = \{\mathcal{A}_A, X\}_{\mathcal{A}} \end{matrix}$$

and more generally for $f: X \to Y$:

$\delta_A f = \{i_A, f\}_{\mathcal{A}\ulcorner}$ = pullback corner map in

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & M_A X \arrow[d] \\ Y_A \arrow[r] & M_A Y \end{tikzcd}$$

Theorem. TFAE

- $\forall A \in \mathcal{A}$: $\delta_A f$ is a (trivial) fibration
- $f \in \operatorname{cocell}\{ \{i^A, p\}_{\ulcorner} \mid A \in \mathcal{A},\ p \in \mathcal{M}$ a (trivial) fibration$\}$

$\rightsquigarrow$ Reedy (trivial) fibrations.

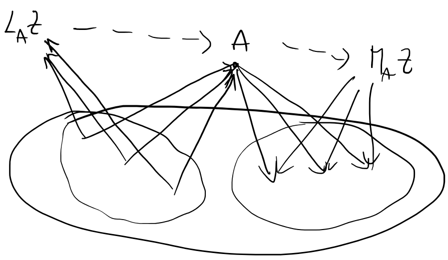
Theorem. There is a model structure on $[\mathcal{A}, \mathcal{M}]$, called the Reedy model structure with $\mathcal{C}$ = Reedy cofibrations, $\mathcal{F}$ = Reedy fibrations, $\mathcal{W}$ = pointwise weak equivalences.

Proof. We need to show that $W \cap \mathcal{C}$ are exactly the Reedy trivial cofibrations. Easily $i_A \,\square\, j \;\square\; \mathcal{F} \iff j \,\square\, \underbrace{\{i_A \,\square\, \mathcal{F}\}_{A}}_{\text{fibrations}}$ by definition.
$\uparrow$ trivial cofibration

$$\begin{tikzcd} L_A X \arrow[r, tail, "\sim"] \arrow[d] & L_A Y \arrow[d] \arrow[dd, bend left] \\ X_A \arrow[r, tail, "\sim"] \arrow[rd] & \bullet \arrow[d, tail, "\sim"] \\ & Y_A \end{tikzcd}$$

The factorizations are produced inductively using the following idea:

$= \partial A^A \otimes_A Z$
$\|$
$\partial A(-, A) \otimes_A Z$



$A_{<n}$

to give an extension of Z from $A_{<n}$ to $A_{\leq n}$ we need $\forall A$ of degree n to factor

$$L_A Z \longrightarrow Z_A \longrightarrow M_A Z$$
$$\underbrace{L_A Z \longrightarrow M_A Z}_{\text{obtained from } Z|_{A_{<n}}}$$

Now

$$\begin{tikzcd} L_A X \arrow[r] \arrow[d] & L_A Z \arrow[r] \arrow[d] & L_A Y \arrow[d] \\ X_A \arrow[r] \arrow[d] & \bullet \arrow[r, tail] \arrow[ul, phantom, "\ulcorner"] & Z_A \arrow[r, two heads, "\sim"] & \bullet \arrow[r] \arrow[d] & Y_A \arrow[d] \\ M_A X \arrow[r] & M_A Z \arrow[r] & M_A Y \end{tikzcd}$$

$\square$

Application. • Properness in $\mathcal{M}_c / \mathcal{M}_f$ differently.

• Given $A \in \mathcal{M}_c$ consider $\mathrm{cst}\, A \in [\Delta, \mathcal{M}]$ and consider its Reedy cofibrant replacement $A^* \in [\Delta, \mathcal{M}]$

$$\begin{tikzcd} A^* \arrow[r] \arrow[rr, bend left] & \bar{A}^* \arrow[r] & \mathrm{cst}\, A \\ \mathrm{sk}_0\, A^* \arrow[u] \arrow[r] & \mathrm{sk}_0\, \mathrm{cst}\, A \arrow[u] \arrow[ul, phantom, "\ulcorner"] \arrow[ur] \end{tikzcd}$$

does the job

$\uparrow$ cofibrant when A is
$(= \Delta_0 \cdot A)$

$$0 \rightarrowtail A^* \xrightarrow{\sim} \mathrm{cst}\, A$$

– we may achieve that $A^*_0 = A$ and then A^* is called a cosimplicial frame on A. Any two are related by a zig-zag of cosimplicial frames. Thus for $X \in \mathcal{M}_f$

$$\mathcal{M}(A^*, X) \in s\mathrm{Set}$$

Dually $\mathcal{M}(A, X_*) \in s\mathrm{Set}$ via a simplicial frame on X.

$\left.\begin{array}{l} A^* : \Delta \longrightarrow \mathcal{M} \\ K : \Delta^{op} \longrightarrow \mathrm{Set} \end{array}\right\}$ get $K \cdot_\Delta A^* \in \mathcal{M}$, an action of $s\mathrm{Set}$ on $\mathcal{M}_c$
$K * A \overset{def}{=} K \cdot_\Delta A^*$, but not associative

The bifunctor $s\mathrm{Set} \times c\mathcal{M} \xrightarrow{- \cdot_\Delta -} \mathcal{M}$ is close to being left Quillen

cofibrations on the left: $i^n \cdot_\Delta \mathcal{C}_R \subseteq \mathcal{C}$
$i^n \cdot_\Delta W \cap \mathcal{C}_R \subseteq W \cap \mathcal{C}$

Reedy cof. between frames

trivial cofibrations on the left more subtle: $(\underbrace{\Delta^{n-1} \overset{\sim}{\rightarrowtail} \Delta^n}) \cdot_\Delta (X \to Y)$:

$$\begin{tikzcd} X_{n-1} \arrow[r] \arrow[d, "\sim"'] & Y_{n-1} \arrow[d] \arrow[dd, bend left, "\sim"] \\ & \bullet \arrow[d] \arrow[ul, phantom, "\ulcorner"] \\ X_n \arrow[r] & Y_n \end{tikzcd}$$

these "generate" anodyne extensions in some sense

$\Delta^0 + \Delta^0 = \partial\Delta^1 \rightarrowtail \Delta^1 \xrightarrow{\sim} \Delta^0$ cylinder on Δ^0

$A + A = \underbrace{\partial\Delta^1 \cdot_\Delta A^*}_{I' A} \rightarrowtail \underbrace{\Delta^1 \cdot_\Delta A^*}_{A^1} \xrightarrow{\sim} A$ cylinder on A

Reflecting... $\mathcal{M}(A^*, X_*) \in s^2\mathrm{Set} = [\Delta^{op} \times \Delta^{op}, \mathrm{Set}]$

Balancing $\quad M(A^*, X_*) \in s^2Set = [\Delta^{op} \times \Delta^{op}, Set]$

comparison maps: $M(A^*, X_*) \longleftarrow M(\mathrm{cst}\, A, X_*)$

$\uparrow$

$M(A^*, X)$

ptwise w.e. between Reedy cof. obj's

$M(A^n, X_*) \xleftarrow{\sim} M(\mathrm{cst}\, A, X_*)$ since $A^n \xrightarrow{\sim} \mathrm{cst}\, A$

Take the geometric realizations:

$$\Delta^\bullet \cdot_\Delta M(A^*, X_*) \xleftarrow{\sim} \Delta^\bullet \cdot_\Delta M(\mathrm{cst}\, A, X_*) = M(A, X_*)$$

$\overset{?}{\|}$

$\mathrm{diag}\, M(A^*, X_*)$

then this + the dual statement proves the balancing

this is about bisimplicial sets $K_{\bullet\bullet} = (K_\bullet)_\bullet$.

$$\Delta^\bullet \cdot_\Delta K_\bullet = \int^{n\in\Delta} \underset{sSet}{\Delta^n} \times \underset{sSet}{K_n} = \int^{n\in\Delta} \underset{sSet}{\Delta^n} \times \left(\int^{k\in\Delta} \underset{sSet}{\Delta^k} \cdot \underset{Set}{K_{nk}}\right)$$

$$= \int^{n,k} \underset{sSet}{(\Delta^n \times \Delta^k)} \cdot \underset{Set}{K_{nk}} = \int^{(n,k)} \Delta\times\Delta(\mathrm{diag}\, -, (n,k)) \cdot K_{nk}$$

$$\overset{\text{Yoneda}}{=} \mathrm{diag}\, K$$

Application $\quad A = (0 \xleftarrow{-} 1 \xrightarrow{+} 2)$.

$$[A, M] \underset{\mathrm{cst}}{\overset{\mathrm{colim}}{\rightleftarrows}} M \qquad \mathrm{cst} \text{ pres w.e. (always) and fibrations: } p: X \twoheadrightarrow Y$$

$\Rightarrow$ colim takes w.e.'s between Reedy cofibrant diagrams to w.e.'s

(would not work for $A = (0 \leftarrow 1 \rightarrow 0)$)

$i_2 \otimes_A p$

$i_1 \otimes_A p$

$i_0 \otimes_A p$

$$\begin{tikzcd} X \arrow[r] & Y \\ X \arrow[u] \arrow[d, "="'] \arrow[r] & Y \arrow[u] \arrow[d, "="] \\ X \arrow[r] & Y \end{tikzcd}$$

Application $\quad M$ simplicial

$$\underset{\substack{\text{proj.}\\ \text{ptwise}}}{[A^{op}, sSet]} \times \underset{\substack{\text{ptwise}\\ \text{proj.}}}{[A, M]} \xrightarrow{-\otimes_A -} M \quad \text{is left Quillen}$$

$Q* = N(-/A)$ a particular cofibrant replacement of $*$

$\Rightarrow Q* \otimes_A - : [A, M_c] \longrightarrow M_c$ preserves w.e.'s

$\downarrow$ is a w.e. on proj. cof. diagrams

$* \otimes_A -$

$\|$

$\underset{A}{\mathrm{colim}}$

$\Rightarrow N(-/A) \otimes_A - = \underset{A}{\mathrm{hocolim}}$

This can be extended to all model categories via frames:

$N(-/\mathcal{A}) : \Delta^{op} \times \mathcal{A}^{op} \longrightarrow Set$

$D^* : \Delta \times \mathcal{A} \longrightarrow \mathcal{M}_c$ frame on $D : \mathcal{A} \to \mathcal{M}_c$

$\rightsquigarrow N(-/\mathcal{A}) \cdot_{\Delta \times \mathcal{A}} D^* =: \underset{\mathcal{A}}{\mathrm{hocolim}}\, D$

$\quad \| $

$N(-/\mathcal{A}) \otimes_{\mathcal{A}} D$ if we denote $K \otimes X = K \cdot_{\Delta} X^*$ the "action" of sSet on $\mathcal{M}$