Weighted colimits

tensor adjunction (or any other)

$\mathrm{Hom}_R(K, \mathrm{Hom}(M,N)) \cong \mathrm{Hom}(K \otimes_R M, N)$

$[R^{op}, Ab]$ $\quad Ab \quad Ab$

$$\underset{\uparrow}{\mathcal{V}}(K, \underset{\uparrow}{\mathcal{M}}(M,N)) \cong \mathcal{M}(K \otimes M, N)$$

replace by diagrams

, e.g. abelian groups and their ⊗

how about $\otimes_R$?

$\bullet_R \longmapsto M$

$R \xrightarrow{\text{ring homo}} \mathrm{End}(M)$

R-module = $R \longrightarrow Ab$ left

$R = \bullet_R$ $\quad R^{op} \longrightarrow Ab$ right

Hom_R = hom in $[R^{op}, Ab]$?

weight $W : A^{op} \longrightarrow \mathcal{V}$ $\quad D : A \longrightarrow \mathcal{M}$ diagram

$$[A^{op}, \mathcal{V}](W, \underbrace{\mathcal{M}(D-, N)}_{A^{op} \xrightarrow{D^{op}} \mathcal{M}^{op} \xrightarrow{\mathcal{M}(-,N)} \mathcal{V}}) \cong \mathcal{M}(W \otimes_A D, N)$$

Definition. The weighted colimit $W \otimes_A D$ ... the colimit of D weighted by W is the object representing $[A^{op}, \mathcal{V}](W, \mathcal{M}(D, -))$ as above.

Examples. • The tensor product of R-modules

- the tensors ... take $A = \bullet_S$ were $S \in \mathcal{V}$ is the unit $\quad W* = K$

$W \otimes_A M = K \otimes M$

- The geometric realization

tensor over Set : $K.P = \coprod_k P$

$$Set(K, Top(P, X)) \cong Top(K \cdot P, X)$$

$$sSet(K_\bullet, \underbrace{Top(|\Delta^\bullet|, X)}_{SX}) \cong Top(\underbrace{K_\bullet \cdot_\Delta |\Delta^\bullet|}_{|K_\bullet|}, X)$$

make Δ - indexed

$\Delta^{op} \xrightarrow{K_\bullet} Set$ weight

$\Delta \xrightarrow[\text{"Yoneda"}]{|\Delta^\bullet|} Top$ diagram

$n \longmapsto |\Delta^n|$

- The ordinary colimits : $\mathcal{V} = Set$ $\quad W = \Delta *$

$$\underbrace{[A^{op}, Set](\Delta *, \mathcal{M}(D, N))}_{cone(D,N)} \cong \mathcal{M}(\Delta * \otimes_A D, N)$$

$$\Delta * (A) : \begin{tikzcd} DA \arrow[r] \arrow[d] & N \arrow[d, equal] \\ DB \arrow[r] & N \end{tikzcd}$$

The general weighted colimits can be translated to ordinary colimits

$$W \otimes_A D = \operatorname*{colim}_{El\, W} DP$$

$El\, W \xrightarrow{P} A \xrightarrow{D} \mathcal{M}$

"take DA multiple times - once for each element of WA"

• A $\quad W_\bullet = K$

$\#K \times \left\{ \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \right.$ $El\, W$

Another point of view :

$\Delta_n = \Delta(n, -)$

$\Delta^n = \Delta(-, n)$

- $[A^{op}, \mathcal{V}](W, \mathcal{M}(D, N)) \cong \mathcal{M}(W \otimes_A D, N)$

take $W = A(-, A) = A^A$ representable functor

$\Rightarrow$ get on the left $\mathcal{M}(D, N)(A) = \mathcal{M}(DA, N)$ by Yoneda

$\mathcal{M}(D-, N)(A)$ $=$

$\Rightarrow A^A \otimes_A D = DA$

- clearly $W \otimes_A D$ preserves colimits (ordinary or weighted) in the W-variable $\Rightarrow$ for $\mathcal{V} = Set$ any weight = presheaf is a colimit of representables $\Rightarrow$ weighted colimits can be replaced by colimits (over the cat of elements)

$$\longrightarrow \mathcal{V} = Ab, \quad A = \begin{tikzcd} a \arrow[loop above, "\mathbb{Z} \cdot 1"] \arrow[r, "\mathbb{Z}"] & b \arrow[loop above, "\mathbb{Z} \cdot 1_b"] \end{tikzcd} \Rightarrow [A, \mathcal{M}] = \text{arrows of } \mathcal{M}$$

$\longrightarrow \quad \mathcal{V} = Ab, \quad \mathcal{A} = \begin{tikzcd} a \arrow[loop above, "\mathbb{Z}\cdot 1"] \arrow[r, "\mathbb{Z}"] \arrow[r, bend right] & b \arrow[loop above, "\mathbb{Z}\cdot 1_b"] \end{tikzcd} \Rightarrow [\mathcal{A}, \mathcal{M}] = $ arrows of $\mathcal{M}$

$\mathcal{A} \to [\mathcal{A}^{op}, Ab]$

$$\begin{tikzcd} a \arrow[d, "f"] \\ b \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}^a \arrow[d, "in"] \\ \mathcal{A}^b \end{tikzcd} \quad \text{colimit} \quad \begin{tikzcd} \mathcal{A}^a \arrow[d, "in"', shift right] \arrow[d, "0", shift left] \\ \mathcal{A}^b \arrow[d] \\ \mathcal{A}^b/\mathcal{A}^a \end{tikzcd}$$

$$\begin{array}{l} \mathcal{A}^a = \mathbb{Z} \leftarrow 0 \\ \mathcal{A}^b = \mathbb{Z} \xleftarrow{1} \mathbb{Z} \\ \hline \mathcal{A}^b/\mathcal{A}^a = 0 \leftarrow \mathbb{Z} \end{array} \quad \rightsquigarrow \quad \begin{array}{l} \mathcal{A}^a \otimes_{\mathcal{A}} D = Da \\ \mathcal{A}^b \otimes_{\mathcal{A}} D = Db \\ \hline \mathcal{A}^b/\mathcal{A}^a \otimes_{\mathcal{A}} D = \operatorname{coker}(Da \to Db) \end{array}$$

coeq's: $\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$

$$\begin{array}{l} \mathcal{A}^a = \{1\} \leftleftarrows \emptyset \\ f_*\downarrow\downarrow g_* \\ \mathcal{A}^b = \{f,g\} \leftleftarrows \{1\} \\ \hline \downarrow \\ \text{coeq} = * \leftleftarrows * \end{array}$$

pushouts: $\begin{tikzcd} a \arrow[r] \arrow[d] & b \\ c \end{tikzcd}$ $\quad \Delta * = \operatorname{colim}(\mathcal{A}^a \rightrightarrows \mathcal{A}^b + \mathcal{A}^c)$

$$\begin{tikzcd} * & * \arrow[l] \\ * \arrow[u] \end{tikzcd} = \operatorname{colim}\left(\begin{tikzcd} * & \emptyset \arrow[l] \\ \emptyset \arrow[u] \end{tikzcd} \rightrightarrows \left(\begin{tikzcd} * & * \arrow[l] \\ \emptyset \arrow[u] \end{tikzcd} + \begin{tikzcd} * & \emptyset \arrow[l] \\ * \arrow[u] \end{tikzcd} \right) \right)$$

- The conical colimits "=" the ordinary colimits in the underlying category $\mathcal{M}_0$:

  $\cdot S: Set \rightleftarrows \mathcal{V}_0 ; \mathcal{V}_0(S, -)$ tensor – hom adjunction

  $\mathbb{Z}: Set \rightleftarrows Ab : Ab_0(\mathbb{Z}, -)$
  free ab. gp

  $\mathcal{A}$ an ordinary category

  $\rightarrow \mathcal{A}\cdot S$ a $\mathcal{V}$-category that admits a canonical weight $\Delta * \cdot S = \Delta S$

$$[\mathcal{A}^{op}, Set](\Delta *, \mathcal{M}_0(D, N)) \cong \mathcal{M}_0(\Delta * \cdot_{\mathcal{A}} D, N)$$
$$\cong$$
$$[\mathcal{A}^{op}, \mathcal{V}_0](\Delta S, \mathcal{M}(D,N))$$
$$\cong$$
$$[(\mathcal{A}\cdot S)^{op}, \mathcal{V}]_0(\Delta S, \mathcal{M}(D,N))$$

this is almost saying that

$$\Delta * \cdot_{\mathcal{A}} D = \Delta S \otimes_{\mathcal{A}S} D$$

up to the index 0

Lemma. If $\mathcal{M}$ has cotensors, this works in the enriched sense

Proof. We need $[(\mathcal{A}\cdot S)^{op}, \mathcal{V}](\Delta S, \mathcal{M}(D,N)) \cong \mathcal{M}(\Delta * \cdot_{\mathcal{A}} D, N)$

$\xLeftrightarrow{\text{Yoneda}} \quad \mathcal{V}_0(K, [(\mathcal{A}\cdot S)^{op}, \mathcal{V}](\Delta S, \mathcal{M}(D,N))) \cong \mathcal{V}_0(K, \mathcal{M}(\Delta * \cdot_{\mathcal{A}} D, N))$

$\overset{\shortparallel}{S \otimes K}$

$$\begin{array}{cc} \| & \| \\ \mathcal{V}_0(S, [(\mathcal{A}\cdot S)^{op}, \mathcal{V}](\Delta S, \mathcal{M}(D, N^K))) & \mathcal{V}_0(S, \mathcal{M}(\Delta * \cdot_{\mathcal{A}} D, N^K)) \\ \| & \| \\ [(\mathcal{A}\cdot S)^{op}, \mathcal{V}]_0(\Delta S, \mathcal{M}(D, N^K)) & \mathcal{M}_0(\Delta * \cdot_{\mathcal{A}} D, N^K) \end{array}$$
$\square$

Summary. ordinary colimits $\overset{\text{often}}{=}$ conical colimits $\subseteq$ weighted colimits $\supseteq$ tensors

Construction. $W \otimes_{\mathcal{A}} D$ can be constructed as the coequalizer of

$$\sum_{B,C} WC \otimes \mathcal{A}(B,C) \otimes DB \underset{1 \otimes act}{\overset{act \otimes 1}{\rightrightarrows}} \sum_{A} WA \otimes DA$$

like $M \otimes_R N$
the coend

$[\mathcal{A}^{op}, \mathcal{V}](F, G)$
$= \int_{\mathcal{A}} \underbrace{\mathcal{V}(F, G)}_{\mathcal{A}^{op} \times \mathcal{A} \to \mathcal{V}}$

Explanation: $[\mathcal{A}^{op}, \mathcal{V}](W, \mathcal{M}(D,N))$ is the equalizer of

$$\prod_{A} \mathcal{V}(WA, \mathcal{M}(DA, N)) \rightrightarrows \prod_{B,C} \{\mathcal{A}(B,C), \mathcal{V}(WC, \mathcal{M}(DB,N))\}$$

$$\begin{tikzcd} WB \arrow[r] & \mathcal{M}B \\ WC \arrow[u] \arrow[r] \arrow[ur] & \mathcal{M}C \arrow[u] \end{tikzcd}$$

$$\begin{array}{cc} \cong & \cong \\ \prod \mathcal{M}(WA \otimes DA, N) & \prod \mathcal{M}(WC \otimes \mathcal{A}(B,C) \otimes DB, N) \end{array}$$

A $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $B\ C$

$\cong$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\cong$

$$\prod_{A} \mathcal{M}(WA \otimes DA, N) \qquad\qquad \prod_{B,C} \mathcal{M}(WC \otimes \mathcal{A}(B,C) \otimes DB, N)$$

obtained from the paraller pair from the construction upon applying $\mathcal{M}(-,N)$ that turns colimits into limits. $\square$

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[r] & \mathrm{colim}\, X \end{tikzcd}$$

Example. $\mathcal{M}$ a simplicial category; then the geometric realization is a functor $|-| : s\mathcal{M} \to \mathcal{M}$ given by

$|X| = \Delta^{\bullet} \otimes_{\Delta} X$, $\Delta^{\bullet} : \Delta \to sSet$ weight, $X : \Delta^{op} \to \mathcal{M}$ diag; the colimit weighted by $\Delta^{\bullet} : \Delta \to sSet$

$\quad$ " Yoneda embedding

We will see that $\Delta^{\bullet}$ is Reedy cofibrant.

The $- \otimes_{\Delta} - : [\Delta, sSet] \times [\Delta^{op}, \mathcal{M}] \to \mathcal{M}$ is left Quillen w.r.t. Reedy structures $\Rightarrow$ $|-|$ homotopy invariant on Reedy cofibrant diag's

E.g. for $\mathcal{M} = sSet \Rightarrow |X| = \mathrm{diag}\, X$

$$|X| = \int^{n} \Delta^{n} \otimes \underbrace{X_n}_{\int^{k} \Delta^{k} \otimes X_{nk}}$$

$$= \int^{n,k} (\Delta^{n} \times \Delta^{k}) \otimes X_{nk}$$

$$= \int^{n,k} \Delta \times \Delta(\mathrm{diag}-,(n,k)) \otimes X_{nk}$$

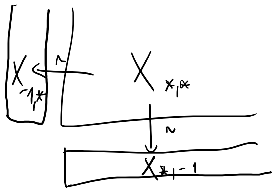
$$= X \circ \mathrm{diag}$$

$$\begin{tikzcd} \Delta^{op} \arrow[r, "\mathrm{diag}"] \arrow[rr, bend right, "X \circ \mathrm{diag}"'] & \Delta^{op} \times \Delta^{op} \arrow[r, "X"] & Set \end{tikzcd}$$

Any X is Reedy cofibrant

If all augmentations $X_n \xrightarrow{\sim} X_{-1}$ are w.e.

$\Rightarrow X \xrightarrow{\sim} \mathrm{cst}\, X_{-1} \Rightarrow |X| \xrightarrow{\sim} |\mathrm{cst}\, X_{-1}| = X_{-1}$



"balancing"

Further interesting topics. Restriction and extension of scalars

$$F_! : [\mathcal{A}, \mathcal{M}] \underset{\perp}{\rightleftarrows} [\mathcal{B}, \mathcal{M}] : F^{*} \qquad\qquad F : \mathcal{A} \to \mathcal{B}$$

$\mathrm{Lan}_F \nearrow$ $\qquad$ $\parallel$

$\mathcal{B}(-,F) \otimes_{\mathcal{A}} - \qquad\qquad \mathcal{B}(F,-) \otimes_{\mathcal{B}} -$

${}_{S}S_{R} \otimes_{R} - \qquad\qquad {}_{R}S_{S} \otimes_{S} -$

Reminder on weighted colimits

$X \times_G Y$

$(xg, y) = (x, gy)$

$$\begin{tikzcd} X \times G \times Y \arrow[d, shift right, "\mathrm{act}_X \times 1"'] \arrow[d, shift left, "1 \times \mathrm{act}_Y"] \\ X \times Y \end{tikzcd}$$

$X : G^{op} \to Set$

$Y : G \to Set$

$M \otimes_R N$

$xr \otimes y = x \otimes ry$

$$\begin{tikzcd} M \otimes R \otimes N \arrow[d, shift right] \arrow[d, shift left] \\ M \otimes N \end{tikzcd}$$

$M : R^{op} \to Ab$

$N : R \to Ab$

$W \otimes_{\mathcal{A}} D$

$$\begin{tikzcd} \sum\limits_{b,c} Wc \otimes \mathcal{A}(b,c) \otimes Db \arrow[d, shift right] \arrow[d, shift left] \\ \sum\limits_{a} Wa \otimes Da \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A}(b,c) \otimes Db \arrow[d] \\ Dc \end{tikzcd} \quad \Bigg| \quad \begin{tikzcd} \mathcal{A}(b,c) \arrow[d, "D"] \\ \mathcal{M}(Db, Dc) \end{tikzcd}$$

$W : \mathcal{A}^{op} \to \mathcal{V}$

$D : \mathcal{A} \to \mathcal{M}$

$\Delta^{\bullet} \times_{\Delta} X$

$(\theta^* t, x) \sim (t, \theta_* x)$

$$\begin{tikzcd} \Delta^n & ---- \arrow[d, shift right] \arrow[d, shift left] & X_n \arrow[d, "d_i"] \\ \Delta^{n-1} \arrow[u, "d^i"] & \sum \Delta^n \times X_n & X_{n-1} \end{tikzcd}$$

$\Delta^{\bullet} : \Delta \to sSet$

$X : \Delta^{op} \to Top \ldots$

$\otimes : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$

$\otimes_{\mathcal{A}} : [\mathcal{A}^{op}, \mathcal{M}] \times [\mathcal{A}, \mathcal{N}] \to \mathcal{P}$

$W \otimes_{\mathcal{A}} D = \int^{A \in \mathcal{A}} WA \otimes DA$

$$\begin{tikzcd} \sum\limits_{b,c} Wc \otimes \mathcal{A}(b,c) \otimes Db \arrow[r, shift left] \arrow[r, shift right] & \sum\limits_{a} Wa \otimes Da \arrow[r, "coend"'] & W \otimes_{\mathcal{A}} D \end{tikzcd} \quad / \ \mathcal{M}(-, N)$$

$$\begin{tikzcd} \prod\limits_{b,c} \{\mathcal{A}(b,c), \mathcal{M}(Wc \otimes Db, N)\} \arrow[d, equal] & \prod\limits_{a} \mathcal{M}(Wa \otimes Da, N) \arrow[l, shift left] \arrow[l, shift right] \arrow[d, equal] & \mathcal{M}(W \otimes_{\mathcal{A}} D, N) \arrow[l] \\ \prod\limits_{b,c} \{\mathcal{A}(b,c), \mathcal{V}(Wc, \mathcal{M}(Db, N))\} & \prod\limits_{a} \mathcal{V}(Wa, \mathcal{M}(Da, N)) \arrow[l, shift left] \arrow[l, shift right] & {[\mathcal{A}^{op}, \mathcal{V}](W, \mathcal{M}(D, N))} \arrow[l] \end{tikzcd}$$

end W-weighted cones $D \Rightarrow N$

$\mathcal{M}(DA, N) \overset{Yoneda}{=} [\mathcal{A}^{op}, \mathcal{V}](\mathcal{A}(-, A), \mathcal{M}(D, N)) \overset{def}{\cong} \mathcal{M}(\mathcal{A}(-, A) \otimes_{\mathcal{A}} D, N)$

Example.

$\mathcal{A}(-, A) \otimes_{\mathcal{A}} D = DA$ dually $\{\mathcal{A}(A, -), D\}_{\mathcal{A}} = DA$

$\mathcal{A}(-, -) \otimes_{\mathcal{A}} D = D \quad \leftarrow \quad R \otimes_R M = M \qquad Hom_R(R, M) = M$

$\mathcal{A}^{op} \to [\mathcal{B}, \mathcal{V}]$

$\left. \begin{array}{l} W : \mathcal{A}^{op} \times \mathcal{B} \to \mathcal{V} \\ D : \mathcal{A} \to \mathcal{M} \end{array} \right\} \quad W \otimes_{\mathcal{A}} D : \mathcal{B} \to \mathcal{M}$

${}_S M_R \otimes_R {}_R N$

Restriction of scalars : $F : \mathcal{B} \to \mathcal{A}$

$\mathcal{A}(-, F-) : \mathcal{A}^{op} \times \mathcal{B} \to \mathcal{V} \qquad D : \mathcal{A} \to \mathcal{M}$

$\mathcal{A}(-, F-) \otimes_{\mathcal{A}} D = DF$ dually $\{\mathcal{A}(F-, -), D\}_{\mathcal{A}} = DF$

$\| $

$F^* D$

R

$R \to S$

${}_{R}\mathbb{C}_{\mathbb{C}} \otimes_{\mathbb{C}} V = V^{R}$

${}_R S_S \otimes_S {}_S M = {}_R M$

Adjunction :

$$[\mathcal{B}, \mathcal{V}](C, \overbrace{\{\mathcal{A}(F-, -), D\}_{\mathcal{A}}}^{F^*D}) = [\mathcal{A}, \mathcal{V}](\underbrace{\mathcal{A}(F-, -) \otimes_{\mathcal{B}} C}, D)$$

$Hom_R(\underset{R}{M}, \underset{S}{Hom_S(S, N)}) = Hom_S(S \otimes_R M, N)$

$F_! C$ extension of scalars (like $S \otimes_R L$)

$[\mathcal{B}, \mathcal{V}](C, F^* D) = [\mathcal{A}, \mathcal{V}](F_! C, D)$

$\leftarrow Lan_F C$ left Kan extension

eg. $F : \mathcal{B} \hookrightarrow \mathcal{A}$ full embedding

$F_! C(B) = \mathcal{A}(F-, B) \otimes C$

e.g. $F : \mathcal{B} \hookrightarrow \mathcal{A}$ full embedding

$$F_! C\,(B) = \mathcal{A}(F-, B) \otimes_{\mathcal{B}} C$$
$$= \mathcal{B}(-, B) \otimes_{\mathcal{B}} C$$
$$= C_B \quad \text{real extension}$$

fully faithful

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "C"] \arrow[d, hook, "F"'] & \mathcal{M} \\
\mathcal{A} \arrow[ur, "\mathrm{Lan}_F C"'] \arrow[u, phantom, shift left=4ex, "\shortparallel"] &
\end{tikzcd}$$

$$\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{D} \mathcal{M} \qquad \mathcal{A} \xrightarrow{D} \mathcal{M} \xrightarrow{1} \mathcal{M} \qquad W \otimes_{\mathcal{A}} D = \ldots \otimes_{\mathcal{M}} 1$$

$$W \otimes_{\mathcal{B}} (D \circ F) = \overline{W} \otimes_{\mathcal{A}} D$$

$$W \otimes_{\mathcal{B}} \big(\underbrace{\mathcal{A}(-, F-) \otimes_{\mathcal{A}} D}_{D \circ F}\big) = \big(\underbrace{W \otimes_{\mathcal{B}} \mathcal{A}(-, F-)}_{\overline{W} = \mathrm{Lan}_F W}\big) \otimes_{\mathcal{A}} D$$

restricted from S to R

$$M \otimes_R N = M \otimes_R (S \otimes_S N)$$
$$= (M \otimes_R S) \otimes_S N$$

Reedy model categories, framings

$\mathcal{M}(M,N)$ $\downarrow(\exists)$

Motivation. Any model category $\mathcal{M}$ is in some weak sense enriched over sSet and, as a result, $Ho(\mathcal{M})$ will be enriched over $Ho(sSet)$.

Start with an honest sSet-model category = simplicial model category

- since $K \in sSet$ is a colimit $K = \operatorname{colim}_{(n,\, \Delta^n \to K)} \Delta^n = K \cdot_\Delta \Delta^\bullet$

$\Delta^\bullet : \Delta \to sSet$
$K : \Delta^{op} \to Set$

we have $K \otimes M = \operatorname{colim}(\Delta^n \otimes M)$ and it is enough to give $\Delta^n \otimes M$

- clearly $\Delta^0 \otimes M \cong M$, since Δ^0 is the monoidal unit
- what is the essential property of $\Delta^1 \otimes M$? it is a cylinder $\overset{?}{}$ … at least for $M \in \mathcal{M}_c$:

$$\Delta^0 + \Delta^0 \xrightarrow[]{[d^0,d^1]} \Delta^1 \xrightarrow[\sim]{s^0} \Delta^0 \quad \rightsquigarrow \quad M + M \rightarrowtail \Delta^1 \otimes M \xrightarrow{\sim} M$$

- what about $\Delta^2 \otimes M$?

$$\begin{tikzcd}
 & & \Delta^0 \arrow[dl] \arrow[dr, "d^0"] & & \\
 & \Delta^1 \arrow[r, "d^1"'] & \Delta^2 & \Delta^1 \arrow[l, "d^0"] & \\
\Delta^0 \arrow[ur] \arrow[rr, "d^1"'] & & \Delta^1 \arrow[u, "d^2"] & & \Delta^0 \arrow[ul, "d^1"'] \arrow[ll, "d^0"]
\end{tikzcd}$$

, more compactly $\partial\Delta^2 \to \Delta^2$
$\parallel$
$L_2\Delta^\bullet$

$\rightsquigarrow L_2\Delta^\bullet \otimes M \rightarrowtail \Delta^2 \otimes M \xrightarrow{\sim} \overset{\mathcal{M}}{\cancel{M_2\Delta^\bullet \otimes}} M$

$$\begin{tikzcd}
 & \Delta^2 \arrow[dl, "s^0"'] \arrow[dr, "s^1"] & \\
\Delta^1 \arrow[dr, "s^0"'] & & \Delta^1 \arrow[dl, "s^0"] \\
 & \Delta^0 &
\end{tikzcd}$$

, more compactly $\Delta^2 \xrightarrow{\sim} M_2\Delta^\bullet$

$\underset{0}{\nabla}$ $\Delta^\bullet : \Delta \longrightarrow sSet$ a cosimplicial object in sSet

- need some calculus of such diagrams $\Delta \to sSet$, $\Delta \to \mathcal{M}$
  → Reedy categories, Reedy model structures

$\rightsquigarrow \Delta^\bullet \otimes M : \Delta \to \mathcal{M}$ a cosimplicial object in $\mathcal{M}$

A Reedy category has two kinds of maps – direct and inverse (like d^i and s^i in Δ)

Definition A direct category is a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A} \to \lambda$ that satisfies $f : A \to B \rightsquigarrow f = 1 \Leftrightarrow \deg A = \deg B$
↑ ordinal
$\Rightarrow \deg A < \deg B$

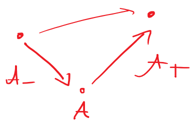
An inverse category is a dual notion, i.e. a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A}^{op} \to \lambda$ satisfying $f = 1 \Leftrightarrow \deg A = \deg B$.

A Reedy category is a category $\mathcal{A}$ together with two subcategories $\mathcal{A}^+$, $\mathcal{A}_-$ and a function $\deg : \operatorname{ob} \mathcal{A} \to \lambda$ that

- makes $\mathcal{A}^+$ into a direct category
- makes $\mathcal{A}_-$ into an inverse category
- any morphism has a unique decomposition:

$$\sum_{A \in \mathcal{A}} \mathcal{A}^+(A,C) \times \mathcal{A}_-(B,A) \xrightarrow[\cong]{\circ} \mathcal{A}(B,C)$$

as a composition of an inverse and a direct morphism



The Yoneda lemma gives $X \cong \mathcal{A}(-,-) \cdot_{\mathcal{A}} X$ and we will describe a way of building X by decomposing $\mathcal{A}(-,-)$. The axioms actually give

$$\mathcal{A}(-,-) = \sum_{A \in \mathcal{A}} \mathcal{A}^+(A,-) \times \mathcal{A}_-(-,A) = \sum_{A \in \mathcal{A}} \mathcal{A}^+_A \times \mathcal{A}^A_-$$

(but only of functors $\mathcal{A}_-^{op} \times \mathcal{A}^+ \to Set$ $\underset{0}{\nabla}$)

but in order to understand $\mathcal{A}(-,-)$ we factor it into

$sk_n\, \mathcal{A}(-,-)$ … maps that factor through some A of degree $\leq n$

but in order to understand $\mathcal{A}(-,-)$ we factor " "

$sk_n\,\mathcal{A}(-,-)$ ... maps that factor through some A of degree $\leq n$

and clearly we have

$$\mathcal{A}(-,-) = \operatorname*{colim}_{n<\lambda} sk_n\,\mathcal{A}(-,-)$$

so that it remains to study the difference between $sk_n\,\mathcal{A}$ and $sk_{n-1}\,\mathcal{A}$ or, better for n limit, $sk_{<n}\,\mathcal{A} = \operatorname*{colim}_{i<n} sk_i\,\mathcal{A}$.

Quite clearly, we have a pushout square

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & sk_{<n}\,\mathcal{A} \arrow[d] \\
\sum\limits_{A\in\mathcal{A}_n} \mathcal{A}^+_A \times \mathcal{A}^A_- \arrow[r] \arrow[ur, phantom, "\ulcorner"] & sk_n\,\mathcal{A}
\end{tikzcd}$$

with A ranging over all objects of degree n

However, it will be crucial to express this in terms of representable functors on $\mathcal{A}$, rather than on $\mathcal{A}^+$, $\mathcal{A}_-$.

Examples.

- any ordinal is a direct category
- $\begin{tikzcd} & \bullet \\ \bullet \arrow[ur, "+"] \arrow[dr, "+"'] & \\ & \bullet \end{tikzcd}$ is a direct category
- $\begin{tikzcd} \bullet & \bullet \arrow[l, "-"'] \arrow[r, "+"] & \bullet \end{tikzcd}$ is a Reedy category
- Δ^+ is a direct category (non-empty finite ordinals + monos)
- Δ_- is an inverse category (non-empty finite ordinals + epis)
- Δ is a Reedy category

$\mathcal{A}$ direct $\Leftrightarrow$ $\mathcal{A}^{op}$ inverse

$\mathcal{A}$ Reedy $\Leftrightarrow$ $\mathcal{A}^{op}$ Reedy

$\Rightarrow$ Δ^{op} Reedy

Notation. We denote

- $\mathcal{A}_A = \mathcal{A}(A,-) \in [\mathcal{A},Set]$ the covariant representable
- $\mathcal{A}^A = \mathcal{A}(-,A) \in [\mathcal{A}^{op},Set]$ the contravariant representable ... think Δ^n

We further define two subfunctors

- $i_A \colon \partial\mathcal{A}_A \subseteq \mathcal{A}_A$ of maps that factor through an object of lower degree.
- $i^A \colon \partial\mathcal{A}^A \subseteq \mathcal{A}^A$ of maps that factor through an object of lower degree.

In the decomposition

$$\mathcal{A}_A = \mathcal{A}(A,-) = \sum \mathcal{A}^+(B,-)\times\mathcal{A}_-(A,B)$$

$$\partial\mathcal{A}_A = \sum \mathcal{A}^+(B,-)\times\partial\mathcal{A}_-(A,B)$$

↖ only 1 is excluded

This means that there is a pushout (coproduct)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \partial\mathcal{A}_A \arrow[d, "i_A"] \\
\mathcal{A}^+_A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{A}_A
\end{tikzcd}$$

Dually, we get

$$\mathcal{A}^A = \mathcal{A}(-,A) = \sum \mathcal{A}^+(B,A)\times\mathcal{A}_-(-,B)$$

$$\partial\mathcal{A}^A = \sum \partial\mathcal{A}^+(B,A)\times\mathcal{A}_-(-,B)$$

and a pushout (coproduct)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \partial\mathcal{A}^A \arrow[d, "i^A"] \\
\mathcal{A}^A_- \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{A}^A
\end{tikzcd}$$

$$\overbrace{\partial\mathcal{A}^+(B,A) \text{ by Yoneda}}$$

$$= \sum \partial(\mathcal{A}^+)^A \times_{\mathcal{A}^+} \mathcal{A}^+(B,-)\times\mathcal{A}_-(-,B)$$

$$= \partial(\mathcal{A}^+)^A \times_{\mathcal{A}^+} \underbrace{\sum \mathcal{A}^+(B,-)\times\mathcal{A}_-(-,B)}_{\mathcal{A}(-,-)}$$

$$= \partial(\mathcal{A}^+)^A \cdot_{\mathcal{A}^+} \mathcal{A}(-,-)$$

$$= \text{extension of } \partial(\mathcal{A}^+)^A$$

$$\Rightarrow \partial\mathcal{A}^A \cdot_{\mathcal{A}} X = (\partial(\mathcal{A}^+)^A \times_{\mathcal{A}^+} \mathcal{A}(-,-)) \cdot_{\mathcal{A}} X$$

$$= \partial(\mathcal{A}^+)^A \cdot_{\mathcal{A}^+} X$$

Putting these together yields that

each square pushout $\Longrightarrow$ gives as a pushout

$$\begin{tikzcd}
& 0\times 0 \arrow[dl] \arrow[dr] & \\
0\times\mathcal{A}^A_- \arrow[dr] & & \mathcal{A}^+_A\times 0 \arrow[dl] \\
& \bullet &
\end{tikzcd}$$

$$\begin{tikzcd}
& \partial\mathcal{A}_A\times\partial\mathcal{A}^A \arrow[dl] \arrow[dr] & \\
\partial\mathcal{A}_A\times\mathcal{A}^A \arrow[dr] \arrow[ddr, bend right] & & \mathcal{A}_A\times\partial\mathcal{A}^A \arrow[dl] \arrow[ddl, bend left] \\
& \bullet \arrow[d, "i_A\times_{\lrcorner} i^A"] & \\
& \mathcal{A}_A\times\mathcal{A}^A &
\end{tikzcd}$$

$$\begin{tikzcd} 0\times A^A_- \arrow[dr] \arrow[ddr, bend right] & & A^+_A\times 0 \arrow[dl] \arrow[ddl, bend left] \\ & \bullet \arrow[d] & \\ & A^+_A\times A^A_- & \end{tikzcd}$$

zero

gives as a pushout

$$\begin{tikzcd} \partial A_A\times A^A \arrow[dr] \arrow[ddr, bend right] & & A_A\times\partial A^A \arrow[dl] \arrow[ddl, bend left] \\ & \bullet \arrow[d, "i_A\times_\lrcorner i^A"] & \\ & A_A\times A^A & \end{tikzcd}$$

Now we get a diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \sum\limits_A \partial A_A\times A^A +_{\partial A_A\cdot\partial A^A} A_A\times\partial A^A \arrow[r] \arrow[d, "\sum\limits_A i_A\times_\lrcorner i^A"] & \mathrm{sk}_{<n}A(-,-) \arrow[d] \\ \sum\limits_A A^+_A\times A^A_- \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \sum\limits_A A_A\times A^A \arrow[r] & \mathrm{sk}_n A(-,-) \end{tikzcd}$$

sums range over $A\in\mathcal{A}$ with $\deg A=n$

in which the outer square is a pushout ⇒ so is the one on the right but now in $[\mathcal{A}^{op}\times\mathcal{A},\mathrm{Set}]_0^\nabla$

Upon applying $-\cdot_{\mathcal{A}}X$, we denote:

$$\begin{tikzcd} \partial A^A\cdot_{\mathcal{A}}X \arrow[r, "i^A\cdot_{\mathcal{A}}X"] \arrow[d, equal, "\mathrm{def}"] & A^A\cdot_{\mathcal{A}}X \arrow[d, equal] \\ L_AX \arrow[r, "\partial_AX"] & X_A \end{tikzcd}$$

latching object

Example. $\mathcal{A}=\Delta^{op}$, $\mathcal{M}=\mathrm{Set}$

$\Rightarrow L_nX\subseteq X_n$ the subset of deg. simplices (needs a bit of work, see above).

Theorem. For any $X\in[\mathcal{A},\mathcal{M}]$ we get $0=\mathrm{sk}_{-1}X$, $X=\mathrm{colim}\ \mathrm{sk}_nX$ and

$$\begin{tikzcd} \sum\limits_A \partial A_A\cdot X_A +_{\partial A_A\cdot L_AX} A_A\cdot L_AX \arrow[r] \arrow[d, "i_A\cdot_\lrcorner\partial_AX"] & \mathrm{sk}_{<n}X \arrow[d] \\ \sum A_A\cdot X_A \arrow[r] & \mathrm{sk}_nX \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

Important special case.

When $\mathcal{A}$ is direct, we have $\partial A_A=0$ and, consequently,

$$\begin{tikzcd} \sum\limits_A A_A\cdot L_AX \arrow[r] \arrow[d] & \mathrm{sk}_{<n}X \arrow[d] \\ \sum\limits_A A_A\cdot X_A \arrow[r] & \mathrm{sk}_nX \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

proj. model structure generated by $A_A\cdot K\to A_A\cdot L$, $A\in\mathcal{A}$, $K\rightarrowtail L$

so that: $\forall A\in\mathcal{A}$: $\partial_AX: L_AX\rightarrowtail X_A$ cofibration

⇒ X is cofibrant in the projective model structure

More generally, for $f:X\to Y$ we denote $\partial_Af=i^A\cdot_{\mathcal{A},\lrcorner}f$, i.e. the pushout corner map in

$$\begin{tikzcd} L_AX \arrow[r] \arrow[d, "\partial_AX"'] & L_AY \arrow[d] \arrow[dr, "\partial_AY"] & \\ X_A \arrow[r] \arrow[rr, bend right] & \bullet \arrow[r, "\partial_Af"] \arrow[ul, phantom, "\ulcorner" very near start] & Y_A \end{tikzcd}$$

$$\partial_Af=\left(\partial A^A\xrightarrow{i^A}A^A\right)\cdot_{\mathcal{A},\lrcorner}\left(X\xrightarrow{f}Y\right)$$

Theorem. For any map $f:X\to Y$ of $[\mathcal{A},\mathcal{M}]$ we get $X=\mathrm{sk}^X_{-1}Y$, $Y=\mathrm{colim}_{n<\lambda}\ \mathrm{sk}^X_nY$ and

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\Sigma i_A\cdot_{\mathcal{A},\lrcorner}\partial_Af"] & \mathrm{sk}^X_{<n}Y \arrow[d] \\ \bullet \arrow[r] & \mathrm{sk}^X_nY \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[d, dashed] \\ \mathrm{sk}_nA(-,-) \arrow[d, dashed] \\ A(-,-) \end{tikzcd}\ \cdot_{\mathcal{A},\lrcorner}\ \begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}\ =\ \begin{tikzcd} X \arrow[d, dashed] \\ \mathrm{sk}^X_nY \arrow[d, dashed] \\ Y \end{tikzcd}\quad \mathrm{sk}^X_nY=\mathrm{sk}_nY+_{\mathrm{sk}_nX}X$$

Theorem. For a map $f:X\to Y$ of $[\mathcal{A},\mathcal{M}]$ TFAE

- $\forall A$: ∂_Af is a (trivial) cofibration
- $f\in\mathrm{cell}\{i_A\cdot_\lrcorner i\mid A\in\mathcal{A},\ i\in\mathcal{M}$ a (trivial) cofibration$\}$

$$\begin{tikzcd} \partial A_A\cdot L+_{\partial A_A\cdot K}A_A\cdot K \arrow[d] \\ A_A\cdot L \end{tikzcd}$$

These maps are called Reedy (trivial) cofibrations.

Proof. The implication "⇓" is the previous theorem.

For the implication "⇑" observe that ∂_A commutes with all colimits constructions so that it is enough to study $\partial_A(i_B \cdot_{\lrcorner} i)$:

$$\begin{array}{c}\partial\mathcal{A}^A\\ \downarrow i^A\\ \mathcal{A}^A\end{array} \star_A \left(\begin{array}{c}\partial\mathcal{A}_B\\ \downarrow i_B\\ \mathcal{A}_B\end{array} \cdot_{\lrcorner} \begin{array}{c}K\\ \downarrow i\\ L\end{array}\right) = \underbrace{\left(\begin{array}{c}\partial\mathcal{A}^A\\ \downarrow i^A\\ \mathcal{A}^A\end{array} \times_A \begin{array}{c}\partial\mathcal{A}_B\\ \downarrow i_B\\ \mathcal{A}_B\end{array}\right)}_{\substack{\text{either iso for } A \neq B\\ \mathcal{A}_{\neq 1}(B,A) \hookrightarrow \mathcal{A}(B,A) \text{ for } A = B\\ \Downarrow\\ \text{pushout of } 0 \to 1}} \cdot_{\lrcorner} \begin{array}{c}K\\ \downarrow i\\ L\end{array}$$

pushout of i

$\square$

Dually, we denote

$$\begin{tikzcd} M_A X \arrow[r, phantom, "="] & \{\partial\mathcal{A}_A, X\}_A \\ X_A \arrow[u, "\delta_A X"] \arrow[r, phantom, "="] & \{\mathcal{A}_A, X\}_A \arrow[u, "{\{i_A, X\}_A}"'] \end{tikzcd}$$

and more generally for $f : X \to Y$:

$\delta_A f = \{i_A, f\}_{A\ulcorner}$ = pullback corner map in

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & M_A X \arrow[d] \\ Y_A \arrow[r] & M_A Y \end{tikzcd}$$

Theorem. TFAE

- $\forall A \in \mathcal{A}$: $\delta_A f$ is a (trivial) fibration
- $f \in \mathrm{cocell}\{\{i^A, p\}_{\ulcorner} \mid A \in \mathcal{A}, p \in \mathcal{M}$ a (trivial) fibration$\}$

⇝ Reedy (trivial) fibrations.

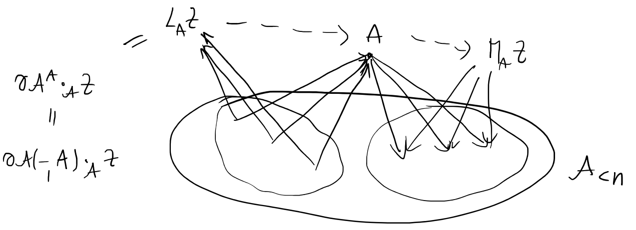
Theorem. There is a model structure on $[\mathcal{A}, \mathcal{M}]$, called the Reedy model structure with $\mathcal{C}$ = Reedy cofibrations, $\mathcal{F}$ = Reedy fibrations, $\mathcal{W}$ = pointwise weak equivalences.

Proof. We need to show that $\mathcal{W} \cap \mathcal{C}$ are exactly the Reedy trivial cofibrations. Easily $\underset{\substack{\uparrow\\ \text{trivial cofibration}}}{i_A \cdot_{\lrcorner} j} \boxslash \mathcal{F} \Leftarrow j \boxslash \underbrace{\{i_A, \mathcal{F}\}_{A\lrcorner}}_{\text{fibrations}}$. by definition

inductively

$$\begin{tikzcd} L_A X \arrow[r, tail, "\sim"] \arrow[d] & L_A Y \arrow[d] \arrow[dr] & \\ X_A \arrow[r, tail, "\sim"] \arrow[rr, bend right] & \bullet \arrow[r, tail] & Y_A \end{tikzcd}$$

The factorizations are produced inductively using the following idea:



to give an extension of Z from $\mathcal{A}_{<n}$ to $\mathcal{A}_{\leq n}$ we need $\forall A$ of degree n to factor

$$\begin{tikzcd} L_A Z \arrow[r] \arrow[rr, bend right, "\text{obtained from } Z|_{\mathcal{A}_{<n}}"'] & Z_A \arrow[r] & M_A Z \end{tikzcd}$$

Now

$$\begin{tikzcd} L_A X \arrow[r] \arrow[d] & L_A Z \arrow[r] \arrow[d] & & L_A Y \arrow[d] \\ X_A \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner"] \arrow[r, tail] & Z_A \arrow[r, two heads, "\sim"] & \bullet \arrow[r] \arrow[dl] \arrow[dl, phantom, "\lrcorner"] & Y_A \arrow[d] \\ M_A X \arrow[from=2-1, to=3-1] \arrow[r] & M_A Z \arrow[rr] & & M_A Y \end{tikzcd}$$

$\square$

Application. • Properness in $\mathcal{M}_c / \mathcal{M}_f$ differently.

Application. • Properness in $\mathcal{M}_c / \mathcal{M}_f$ differently.

• Given $A \in \mathcal{M}_c$ consider $\mathrm{cst}\, A \in [\Delta, \mathcal{M}]$ and consider its Reedy cofibrant replacement $A^* \in [\Delta, \mathcal{M}]$

$$0 \rightarrowtail A^* \xrightarrow{\sim} \mathrm{cst}\, A$$

$$\begin{tikzcd} A^* \arrow[r] \arrow[rr, bend left] & \bar{A}^* \arrow[r] & \mathrm{cst}\, A \\ \mathrm{sk}_0\, A^* \arrow[u] \arrow[rr] & & \mathrm{sk}_0\, \mathrm{cst}\, A \arrow[ul] \arrow[u] \end{tikzcd}$$

does the job

cofibrant when A is $(= \Delta_0 \cdot A)$

– we may achieve that $A^*_0 = A$ and then A^* is called a cosimplicial frame on A. Any two are related by a zig-zag of cosimplicial frames. Thus for $X \in \mathcal{M}_f$

$$\mathcal{M}(A^*, X) \in \mathrm{sSet}$$

Dually $\mathcal{M}(A, X_*) \in \mathrm{sSet}$ via a simplicial frame on X.

$$\left.\begin{array}{l} A^* : \Delta \longrightarrow \mathcal{M} \\ K : \Delta^{op} \longrightarrow \mathrm{Set} \end{array}\right\} \text{ get } K \cdot_\Delta A^* \in \mathcal{M},$$

an action of sSet on $\mathcal{M}_c$ $K * A \overset{\text{def}}{=} K \cdot_\Delta A^*$, but not associative

The bifunctor $\mathrm{sSet} \times c\mathcal{M} \xrightarrow{- \cdot_\Delta -} \mathcal{M}$ is close to being left Quillen

cofibrations on the left: $i^n \cdot_{\Delta} \mathcal{C}_R \subseteq \mathcal{C}$

$i^n \cdot_{\Delta} \mathcal{W} \cap \mathcal{C}_e \subseteq \mathcal{W} \cap \mathcal{C}$

trivial cofibrations on the left more subtle: $(\underbrace{\Delta^{n-1} \overset{\sim}{\rightarrowtail} \Delta^n}_{}) \cdot_\Delta (X \rightarrow Y)$:

Reedy cof. between frames

$$\begin{tikzcd} X_{n-1} \arrow[r] \arrow[d, tail, "\sim"'] & Y_{n-1} \arrow[d, tail, "\sim"] \arrow[dd, tail, bend left, "\sim"] \\ X_n \arrow[r] \arrow[dr] & {} \arrow[d] \\ & Y_n \end{tikzcd}$$

these "generate" anodyne extensions in some sense

$\Delta^0 + \Delta^0 = \partial\Delta^1 \rightarrowtail \Delta^1 \xrightarrow{\sim} \Delta^0$ cylinder on Δ^0

$A + A = \underbrace{\partial\Delta^1 \cdot_\Delta A^*}_{L^1 A} \rightarrowtail \underbrace{\Delta^1 \cdot_\Delta A^*}_{A^1} \xrightarrow{\sim} A$ cylinder on A

Balancing. $\mathcal{M}(A^*, X_*) \in s^2\mathrm{Set} = [\Delta^{op} \times \Delta^{op}, \mathrm{Set}]$

comparison maps:

$$\begin{tikzcd} \mathcal{M}(A^*, X_*) & \mathcal{M}(\mathrm{cst}\, A, X_*) \arrow[l] \\ \mathcal{M}(A^*, X) \arrow[u] & \end{tikzcd}$$

ptwise w.e. between Reedy cof. obj's

$\mathcal{M}(A^n, X_*) \xleftarrow{\sim} \mathcal{M}(\mathrm{cst} A, X_*)$ since $A^n \xrightarrow{\sim} \mathrm{cst}\, A$

Take the geometric realizations:

$$\Delta^\bullet \cdot_\Delta \mathcal{M}(A^*, X_*) \xleftarrow{\sim} \Delta^\bullet \cdot_\Delta \mathcal{M}(\mathrm{cst}\, A, X_*) = \mathcal{M}(A, X_*)$$

$$\overset{?}{\|}$$

$$\mathrm{diag}\, \mathcal{M}(A^*, X_*)$$

then this + the dual statement proves the balancing

this is about bisimplicial sets $K_{\bullet\bullet} = (K_\bullet)_\bullet$.

$$\Delta^\bullet \cdot_\Delta K_\bullet = \int^{n \in \Delta} \underset{\mathrm{sSet}}{\Delta^n} \times \underset{\mathrm{sSet}}{K_n} = \int^{n \in \Delta} \underset{\mathrm{sSet}}{\Delta^n} \times \Big(\int^{k \in \Delta} \underset{\mathrm{sSet}}{\Delta^k} \cdot \underset{\mathrm{Set}}{K_{nk}} \Big)$$

$$= \int^{n,k} \underset{\mathrm{sSet}}{(\Delta^n \times \Delta^k)} \cdot \underset{\mathrm{Set}}{K_{nk}} = \int^{(n,k)} \Delta \times \Delta (\mathrm{diag}\, -, (n,k)) \cdot K_{nk}$$

$$\overset{\text{Yoneda}}{=} \mathrm{diag}\, K$$

Application. $A = (0 \xleftarrow{-} 1 \xrightarrow{+} 2)$.

$$[A, \mathcal{M}] \underset{\mathrm{cst}}{\overset{\mathrm{colim}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{M}$$

cst pres w.e. (always) and fibrations: $p : X \twoheadrightarrow Y$

$\Rightarrow$ colim takes w.e.'s between Reedy cofibrant

$i_2 \cdot_A p$

$$\begin{tikzcd} X \arrow[r] & Y \\ X_{\cdot\cdot} \arrow[u] \arrow[r] \arrow[ur] & Y \arrow[u, hook] \end{tikzcd}$$

⇒ colim takes w.e.'s between Reedy cofibrant diagrams to w.e.'s

$$\begin{tikzcd}
X \arrow[rr, "i_2 \cdot_{\mathcal{A}} P"] & & Y \\
X \arrow[u, hook] \arrow[r, "i_1 \cdot_{\mathcal{A}} P"] \arrow[rr, bend left] \arrow[d] & \cdot \arrow[r] \arrow[d, "="] \arrow[dl, phantom, "\lrcorner"] & Y \arrow[u, hook] \arrow[d, "="] \\
X \arrow[rr, "i_0 \cdot_{\mathcal{A}} P"] & & Y
\end{tikzcd}$$

(would not work for
$\mathcal{A} = (0 \leftleftarrows 1 \rightrightarrows 0)$)

Application. $\mathcal{M}$ simplicial

$$[\mathcal{A}^{op}, sSet] \times [\mathcal{A}, \mathcal{M}] \xrightarrow{-\otimes_{\mathcal{A}}-} \mathcal{M}$$ is left Quillen

proj. / ptwise   ptwise / proj.

$Q* = N(-/\mathcal{A})$ a particular cofibrant replacement of $*$

⇒ $Q* \otimes_{\mathcal{A}} - : [\mathcal{A}, \mathcal{M}_c] \longrightarrow \mathcal{M}_c$ preserves w.e.'s

$\downarrow$ is a w.e. on proj. cof. diagrams

$* \otimes_{\mathcal{A}} -$

$\parallel$

$\operatorname{colim}_{\mathcal{A}}$

⇒ $N(-/\mathcal{A}) \otimes_{\mathcal{A}} - = \operatorname{hocolim}_{\mathcal{A}}$

This can be extended to all model categories via frames:

$N(-/\mathcal{A}) : \Delta^{op} \times \mathcal{A}^{op} \longrightarrow Set$

$D^* : \Delta \times \mathcal{A} \longrightarrow \mathcal{M}_c$ frame on $D : \mathcal{A} \to \mathcal{M}_c$

⇝ $N(-/\mathcal{A}) \cdot_{\Delta \times \mathcal{A}} D^* =: \operatorname{hocolim}_{\mathcal{A}} D$

$\parallel$

$N(-/\mathcal{A}) \otimes_{\mathcal{A}} D$ if we denote $K \otimes X = K \cdot_{\Delta} X^*$ the "action" of sSet on $\mathcal{M}$