Reminder on weighted colimits

$X \times_G Y$

$(xg, y) = (x, gy)$

$$\begin{tikzcd} X \times G \times Y \arrow[d, shift right, "\mathrm{act}_X \times 1"'] \arrow[d, shift left, "1 \times \mathrm{act}_Y"] \\ X \times Y \end{tikzcd}$$

$X : G^{op} \to \mathrm{Set}$

$Y : G \to \mathrm{Set}$

$M \otimes_R N$

$xr \otimes y = x \otimes ry$

$$\begin{tikzcd} M \otimes R \otimes N \arrow[d, shift right] \arrow[d, shift left] \\ M \otimes N \end{tikzcd}$$

$M : R^{op} \to Ab$

$N : R \to Ab$

$W \otimes_{\mathcal{A}} D$

$$\begin{tikzcd} \sum_{b,c} Wc \otimes \mathcal{A}(b,c) \otimes Db \arrow[d, shift right] \arrow[d, shift left] \\ \sum_a Wa \otimes Da \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A}(b,c) \otimes Db \arrow[d] \\ Dc \end{tikzcd} \quad \Bigg| \quad \begin{tikzcd} \mathcal{A}(b,c) \arrow[d, "D"] \\ \mathcal{M}(Db, Dc) \end{tikzcd}$$

$W : \mathcal{A}^{op} \to \mathcal{V}$

$D : \mathcal{A} \to \mathcal{M}$

$\otimes : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$

$\otimes_{\mathcal{A}} : [\mathcal{A}^{op}, \mathcal{M}] \times [\mathcal{A}, \mathcal{N}] \to \mathcal{P}$

$\Delta^{\cdot} \times_{\Delta} X$

$(\theta^* t, x) \sim (t, \theta_* x)$

$$\begin{tikzcd} \Delta^n & \text{- - - -} \arrow[d, shift right] \arrow[d, shift left] & X_n \arrow[d, "d_i"] \\ \Delta^{n-1} \arrow[u, "d^i"'] & \sum \Delta^n \times X_n & X_{n-1} \end{tikzcd}$$

$\Delta^{\cdot} : \Delta \to sSet$

$X : \Delta^{op} \to Top \; \dots$

$W \otimes_{\mathcal{A}} D = \int^{A \in \mathcal{A}} WA \otimes DA$

$$\begin{tikzcd} \sum_{b,c} Wc \otimes \mathcal{A}(b,c) \otimes Db \arrow[r, shift left] \arrow[r, shift right] & \sum_a Wa \otimes Da \arrow[r, "\mathrm{coend}"'] & W \otimes_{\mathcal{A}} D \end{tikzcd} \quad / \; \mathcal{M}(-, N)$$

$$\begin{tikzcd} \prod_{b,c} \{\mathcal{A}(b,c), \mathcal{M}(Wc \otimes Db, N)\} \arrow[d, equal] & \prod_a \mathcal{M}(Wa \otimes Da, N) \arrow[l, shift left] \arrow[l, shift right] \arrow[d, equal] & \mathcal{M}(W \otimes_{\mathcal{A}} D, N) \arrow[l] \\ \prod_{b,c} \{\mathcal{A}(b,c), \mathcal{V}(Wc, \mathcal{M}(Db, N))\} & \prod_a \mathcal{V}(Wa, \mathcal{M}(Da, N)) \arrow[l, shift left] \arrow[l, shift right] & {[\mathcal{A}^{op}, \mathcal{V}]}(W, \mathcal{M}(D, N)) \arrow[l, "\mathrm{end}"'] \end{tikzcd}$$

W-weighted cones $D \Rrightarrow N$

$\mathcal{M}(DA, N) \overset{\text{Yoneda}}{=} [\mathcal{A}^{op}, \mathcal{V}](\mathcal{A}(-, A), \mathcal{M}(D, N)) \overset{\text{def}}{\cong} \mathcal{M}(\mathcal{A}(-, A) \otimes_{\mathcal{A}} D, N)$

Example. $\mathcal{A}(-, A) \otimes_{\mathcal{A}} D = DA$ dually $\{\mathcal{A}(A, -), D\}_{\mathcal{A}} = DA$

$\mathcal{A}(-, -) \otimes_{\mathcal{A}} D = D \quad \leftarrow \quad R \otimes_R M = M \qquad \mathrm{Hom}_R(R, M) = M$

$\mathcal{A}^{op} \to [\mathcal{B}, \mathcal{V}]$

$\left.\begin{array}{l} W : \mathcal{A}^{op} \times \mathcal{B} \to \mathcal{V} \\ D : \mathcal{A} \to \mathcal{M} \end{array}\right\} \quad W \otimes_{\mathcal{A}} D : \mathcal{B} \to \mathcal{M}$

${}_S M_R \otimes_R {}_R N$

Restriction of scalars: $F : \mathcal{B} \to \mathcal{A}$

$\mathcal{A}(-, F-) : \mathcal{A}^{op} \times \mathcal{B} \to \mathcal{V}$ $\quad D : \mathcal{A} \to \mathcal{M}$

$\mathcal{A}(-, F-) \otimes_{\mathcal{A}} D = DF$ $\quad$ dually $\{\mathcal{A}(F-, -), D\}_{\mathcal{A}} = DF$

$\| $

$F^* D$

$R \qquad R \to S \qquad {}_R S_S \otimes_S V = V^R$

${}_R S_S \otimes_S {}_S M = {}_R M$

Adjunction:

$[\mathcal{B}, \mathcal{V}](C, \overbrace{\{\mathcal{A}(F-, -), D\}_{\mathcal{A}}}^{F^*D}) = [\mathcal{A}, \mathcal{V}](\underbrace{\mathcal{A}(F-, -) \otimes_{\mathcal{B}} C}_{F_! C}, D)$ $\quad$ extension of scalars (like $S \otimes_R L$)

$\mathrm{Hom}_R(\underset{R}{M}, \mathrm{Hom}_S(\underset{S}{S}, N)) = \mathrm{Hom}_S(S \otimes_R M, N)$

$[\mathcal{B}, \mathcal{V}](C, F^* D) = [\mathcal{A}, \mathcal{V}](F_! C, D)$

$\mapsto \mathrm{Lan}_F C$ $\quad$ left Kan extension

eg. $F : \mathcal{B} \hookrightarrow \mathcal{A}$ full embedding

$F_! C(B) = \mathcal{A}(F-, B) \otimes C$

fully faithful

$$\begin{tikzcd} \mathcal{B} \arrow[r, "C"] \arrow[d, hook, "F"'] & \mathcal{M} \\ \mathcal{A} \arrow[ur, "\mathrm{Lan}_F C"'] \arrow[u, phantom, "//"] & \end{tikzcd}$$

eg. $F : \mathcal{B} \hookrightarrow \mathcal{A}$ full embedding

$$F_! C\,(B) = \mathcal{A}(F-, B) \otimes_{\mathcal{B}} C$$

$$= \mathcal{B}(-, B) \otimes_{\mathcal{B}} C$$

$$= C_B \qquad \text{real extension}$$

$$\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{D} \mathcal{M} \qquad \mathcal{A} \xrightarrow{D} \mathcal{M} \xrightarrow{1} \mathcal{M} \qquad W \otimes_{\mathcal{A}} D = \ldots \otimes_{\mathcal{M}} 1$$

$$W \otimes_{\mathcal{B}} (D \circ F) = \overline{W} \otimes_{\mathcal{A}} D$$

$$W \otimes_{\mathcal{B}} \Big(\underbrace{\mathcal{A}(-, F-) \otimes_{\mathcal{A}} D}_{D \circ F} \Big) = \Big(\underbrace{W \otimes_{\mathcal{B}} \mathcal{A}(-, F-)}_{\overline{W} = \mathrm{Lan}_F W} \Big) \otimes_{\mathcal{A}} D$$

restricted from S to R

$$M \otimes_R N = M \otimes_R (S \otimes_S N)$$

$$= (M \otimes_R S) \otimes_S N$$

Reedy model categories, framings

$\mathcal{M}(M,N)$ $\overset{!}{\downarrow}\exists$

Motivation. Any model category $\mathcal{M}$ is in some weak sense enriched over sSet and, as a result, $Ho(\mathcal{M})$ will be enriched over $Ho(sSet)$.

Start with an honest sSet-model category = simplicial model category

- since $K \in sSet$ is a colimit $K = \operatorname*{colim}_{(n,\, \Delta^n \to K)} \Delta^n = K \cdot_\Delta \Delta^\bullet$

$\Delta^\bullet : \Delta \to sSet$
$K : \Delta^{op} \to Set$

we have $K \otimes M = \operatorname{colim}(\Delta^n \otimes M)$ and it is enough to give $\Delta^n \otimes M$

- clearly $\Delta^0 \otimes M \cong M$, since Δ^0 is the monoidal unit
- what is the essential property of $\Delta^1 \otimes M$?

  it is a cylinder ... at least for $M \in \mathcal{M}_c$:

$$\Delta^0 + \Delta^0 \xrightarrow{[d^0,d^1]} \Delta^1 \xrightarrow[\sim]{s^0} \Delta^0 \quad\rightsquigarrow\quad M + M \rightarrowtail \Delta^1 \otimes M \xrightarrow{\sim} M$$

- what about $\Delta^2 \otimes M$?

$$\begin{tikzcd} & \Delta^0 \arrow[dl] \arrow[dr, "d^0"] & \\ \Delta^1 \arrow[r, "d^1"] & \Delta^2 & \Delta^1 \arrow[l, "d^0"'] \\ \Delta^0 \arrow[u] \arrow[r, "d^1"'] & \Delta^1 \arrow[u, "d^2"] & \Delta^0 \arrow[u, "d^1"'] \arrow[l, "d^0"] \end{tikzcd}$$

, more compactly $\partial\Delta^2 \to \Delta^2$, $\partial\Delta^2 = L_2\Delta^\bullet$

$\rightsquigarrow L_2\Delta^\bullet \otimes M \rightarrowtail \Delta^2 \otimes M \xrightarrow{\sim} \overset{\mathcal{M}}{\cancel{M_2\Delta^\bullet\otimes}}M$

$$\begin{tikzcd} & \Delta^2 \arrow[dl, "s^0"'] \arrow[dr, "s^1"] & \\ \Delta^1 \arrow[dr, "s^0"'] & & \Delta^1 \arrow[dl, "s^0"] \\ & \Delta^0 & \end{tikzcd}$$

, more compactly $\Delta^2 \xrightarrow{\sim} M_2\Delta^\bullet$

$\underset{0}{\forall}$ $\Delta^\bullet : \Delta \longrightarrow sSet$
a cosimplicial object in sSet

- need some calculus of such diagrams $\Delta \to sSet$ $\rightsquigarrow$ $\Delta^\bullet \otimes M : \Delta \to \mathcal{M}$
  a cosimplicial object in $\mathcal{M}$

  $\to$ Reedy categories, Reedy model structures $\Delta \to \mathcal{M}$

A Reedy category has two kinds of maps – direct and inverse (like d^i and s^i in Δ)

Definition A direct category is a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A} \longrightarrow \lambda$ that satisfies $f : A \to B \rightsquigarrow f = 1 \iff \deg A = \deg B$
λ ↑ ordinal $\Rightarrow \deg A < \deg B$

An inverse category is a dual notion, i.e. a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A}^{op} \longrightarrow \lambda$ satisfying $f = 1 \iff \deg A = \deg B$.

A Reedy category is a category $\mathcal{A}$ together with two subcategories $\mathcal{A}^+$, $\mathcal{A}_-$ and a function $\deg : \operatorname{ob}\mathcal{A} \to \lambda$ that

- makes $\mathcal{A}^+$ into a direct category
- makes $\mathcal{A}_-$ into an inverse category
- any morphism has a unique decomposition:

$$\begin{tikzcd} \bullet \arrow[rr] \arrow[dr, "\mathcal{A}_-"'] & & \bullet \\ & A \arrow[ur, "\mathcal{A}_+"'] & \end{tikzcd}$$

$$\sum_{A\in\mathcal{A}} \mathcal{A}^+(A,C) \times \mathcal{A}_-(B,A) \xrightarrow[\cong]{\circ} \mathcal{A}(B,C)$$

- any morphism

$$\sum_{A\in\mathcal{A}} \mathcal{A}^+(A,C)\times\mathcal{A}_-(B,A) \xrightarrow[\cong]{\circ} \mathcal{A}(B,C)$$

$\mathcal{A}$

as a composition of an inverse and a direct morphism

The Yoneda lemma gives $X \cong \mathcal{A}(-,-)\cdot_{\mathcal{A}} X$ and we will describe a way of building X by decomposing $\mathcal{A}(-,-)$. The axioms actually give

$$\mathcal{A}(-,-) = \sum_{A\in\mathcal{A}} \mathcal{A}^+(A,-)\times\mathcal{A}_-(-,A) = \sum_{A\in\mathcal{A}} \mathcal{A}^+_A \times \mathcal{A}^A_-$$

(but only of functors $\mathcal{A}_-^{op}\times\mathcal{A}^+\to Set$!)

but in order to understand $\mathcal{A}(-,-)$ we factor it into

$sk_n\,\mathcal{A}(-,-)$ ... maps that factor through some A of degree $\leq n$

and clearly we have

$$\mathcal{A}(-,-) = \operatorname*{colim}_{n<\lambda} sk_n\,\mathcal{A}(-,-)$$

so that it remains to study the difference between $sk_n\mathcal{A}$ and $sk_{n-1}\mathcal{A}$ or, better for n limit, $sk_{<n}\mathcal{A} = \operatorname*{colim}_{i<n} sk_i\,\mathcal{A}$.

Quite clearly, we have a pushout square

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & sk_{<n}\mathcal{A} \arrow[d] \\
\sum_{A\in\mathcal{A}_n} \mathcal{A}^+_A\times\mathcal{A}^A_- \arrow[r] & sk_n\,\mathcal{A} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

with A ranging over all objects of degree n

However, it will be crucial to express this in terms of representable functors on $\mathcal{A}$, rather than on $\mathcal{A}^+$, $\mathcal{A}_-$.

Examples.

$\mathcal{A}$ direct $\iff$ $\mathcal{A}^{op}$ inverse

$\mathcal{A}$ Reedy $\iff$ $\mathcal{A}^{op}$ Reedy

- any ordinal is a direct category
- $\bullet \overset{+}{\longrightarrow} \bullet$, $\bullet \overset{+}{\longrightarrow} \bullet$ (both from one point) is a direct category
- $\bullet \overset{-}{\longleftarrow} \bullet \overset{+}{\longrightarrow} \bullet$ is a Reedy category
- Δ^+ is a direct category (non-empty finite ordinals + monos)
- Δ_- is an inverse category (non-empty finite ordinals + epis)
- Δ is a Reedy category $\Rightarrow$ Δ^{op} Reedy

Notation. We denote

- $\mathcal{A}_A = \mathcal{A}(A,-) \in [\mathcal{A}, Set]$ the covariant representable
- $\mathcal{A}^A = \mathcal{A}(-,A) \in [\mathcal{A}^{op}, Set]$ the contravariant representable ... think Δ^n

We further define two subfunctors

- $i_A\colon \partial\mathcal{A}_A \subseteq \mathcal{A}_A$ of maps that factor through an object of lower degree.
- $i^A\colon \partial\mathcal{A}^A \subseteq \mathcal{A}^A$ of maps that factor through an object of lower degree.

In the decomposition

$$\mathcal{A}_A = \mathcal{A}(A,-) = \sum \mathcal{A}^+(B,-)\times\mathcal{A}_-(A,B)$$

$$\partial\mathcal{A}_A = \sum \mathcal{A}^+(B,-)\times\partial\mathcal{A}_-(A,B)$$

↑ only 1 is excluded

This means that there is a pushout (coproduct)

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \partial \mathcal{A}_A \arrow[d, "i_A"] \\ \mathcal{A}^+_A \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{A}_A \end{tikzcd}$$

Dually, we get

$$\mathcal{A}^A = \mathcal{A}(-, A) = \sum \mathcal{A}^+(B, A) \times \mathcal{A}_-(-, B)$$

$$\partial \mathcal{A}^A = \sum \partial \mathcal{A}^+(B, A) \times \mathcal{A}_-(-, B)$$

$$= \sum \overbrace{\partial(\mathcal{A}^+)^A \times_{\mathcal{A}^+} \mathcal{A}^+(B, -)}^{\partial \mathcal{A}^+(B, A) \text{ by Yoneda}} \times \mathcal{A}_-(-, B)$$

$$= \partial(\mathcal{A}^+)^A \times_{\mathcal{A}^+} \underbrace{\sum \mathcal{A}^+(B, -) \times \mathcal{A}_-(-, B)}_{\mathcal{A}(-,-)}$$

$$= \partial(\mathcal{A}^+)^A \cdot_{\mathcal{A}^+} \mathcal{A}(-,-)$$

$$= \text{extension of } \partial(\mathcal{A}^+)^A$$

$$\Rightarrow \partial \mathcal{A}^A \cdot_{\mathcal{A}} X = (\partial(\mathcal{A}^+)^A \times_{\mathcal{A}^+} \mathcal{A}(-,-)) \cdot_{\mathcal{A}} X = \partial(\mathcal{A}^+)^A \cdot_{\mathcal{A}^+} X$$

(coproduct) and a pushout

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \partial \mathcal{A}^A \arrow[d, "i^A"] \\ \mathcal{A}^A_- \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{A}^A \end{tikzcd}$$

Putting these together yields that

$$\begin{tikzcd} & 0 \times 0 \arrow[dl] \arrow[dr] & \\ 0 \times \mathcal{A}^A_- \arrow[r] \arrow[dr] & \bullet \arrow[d] & \mathcal{A}^+_A \times 0 \arrow[l] \arrow[dl] \\ & \mathcal{A}^+_A \times \mathcal{A}^A_- & \end{tikzcd}$$

zero

each square pushout $\Longrightarrow$ gives as a pushout

$$\begin{tikzcd} & \partial \mathcal{A}_A \times \partial \mathcal{A}^A \arrow[dl] \arrow[dr] & \\ \partial \mathcal{A}_A \times \mathcal{A}^A \arrow[r] \arrow[dr] & \bullet \arrow[d, "i_A \times^{\lrcorner} i^A"] & \mathcal{A}_A \times \partial \mathcal{A}^A \arrow[l] \arrow[dl] \\ & \mathcal{A}_A \times \mathcal{A}^A & \end{tikzcd}$$

Now we get a diagram

$$\begin{tikzcd}[column sep=large] 0 \arrow[r] \arrow[d] & \sum\limits_A \partial \mathcal{A}_A \times \mathcal{A}^A \underset{\partial \mathcal{A}_A \times \partial \mathcal{A}^A}{+} \mathcal{A}_A \times \partial \mathcal{A}^A \arrow[r] \arrow[d, "\sum\limits_A i_A \times^{\lrcorner} i^A"] & \mathrm{sk}_{<n} \mathcal{A}(-,-) \arrow[d] \\ \sum\limits_A \mathcal{A}^+_A \times \mathcal{A}^A_- \arrow[r] & \sum\limits_A \mathcal{A}_A \times \mathcal{A}^A \arrow[r] \arrow[ur, phantom, "\ulcorner", very near start] & \mathrm{sk}_n \mathcal{A}(-,-) \end{tikzcd}$$

sums range over $A \in \mathcal{A}$ with $\deg A = n$

in which the outer square is a pushout $\Rightarrow$ so is the one on the right but now in $[\mathcal{A}^{op} \times \mathcal{A}, \mathrm{Set}]$ $\triangledown$

Upon applying $- \cdot_{\mathcal{A}} X$, we denote:

$$\begin{tikzcd} \partial \mathcal{A}^A \cdot_{\mathcal{A}} X \arrow[r, "i^A \cdot_{\mathcal{A}} X"] \arrow[d, equal, "\mathrm{def}"] & \mathcal{A}^A \cdot_{\mathcal{A}} X \arrow[d, equal] \\ L_A X \arrow[r, "\partial_A X"] & X_A \end{tikzcd}$$

latching object

Example. $\mathcal{A} = \Delta^{op}$, $\mathcal{M} = \mathrm{Set}$

$\Rightarrow L_n X \subseteq X_n$ the subset of deg. simplices (needs a bit of work, see above).

Theorem. For any $X \in [\mathcal{A}, \mathcal{M}]$ we get $0 = \mathrm{sk}_{-1} X$, $X = \mathrm{colim}\ \mathrm{sk}_n X$ and

$$\begin{tikzcd} \sum\limits_A \partial \mathcal{A}_A \cdot X_A \underset{\partial \mathcal{A}_A \cdot L_A X}{+} \mathcal{A}_A \cdot L_A X \arrow[r] \arrow[d, "i_A \cdot^{\lrcorner} \partial_A X"] & \mathrm{sk}_{<n} X \arrow[d] \\ \sum \mathcal{A}_A \cdot X_A \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{sk}_n X \end{tikzcd}$$

Important special case.

When $\mathcal{A}$ is direct, we have $\partial \mathcal{A}_A = 0$ and, consequently,

$$\begin{tikzcd} \sum\limits_A \mathcal{A}_A \cdot L_A X \arrow[r] \arrow[d] & \mathrm{sk}_{<n} X \arrow[d] \\ {} \arrow[ur, phantom, "\ulcorner", very near end] & {} \end{tikzcd}$$

proj. model structure generated by

$$\begin{tikzcd} \sum_{A} \mathcal{A}_{A} \cdot L_A X \arrow[r] \arrow[d] & \mathrm{sk}_{<n} \wedge \arrow[d] \\ \sum_{\mathcal{A}} \mathcal{A}_{\mathcal{A}} \cdot X_{\mathcal{A}} \arrow[r] & \mathrm{sk}_n X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

proj. model structure generated by

$\mathcal{A}_{\mathcal{A}} \cdot K \longrightarrow \mathcal{A}_{\mathcal{A}} \cdot L$

$A \in \mathcal{A}$, $K \rightarrowtail L$

so that: $\forall A \in \mathcal{A}$: $\partial_A X \colon L_A X \rightarrowtail X_{\mathcal{A}}$ cofibration

$\Rightarrow$ X is cofibrant in the projective model structure

More generally, for $f \colon X \longrightarrow Y$ we denote $\partial_A f = i^A \cdot_{\mathcal{A}\lrcorner} f$, i.e. the pushout corner map in

$$\begin{tikzcd} L_A X \arrow[r] \arrow[d, "\partial_A X"'] & L_A Y \arrow[d] \arrow[ddr, bend left, "\partial_A Y"] & \\ X_A \arrow[r] \arrow[drr, bend right] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\partial_A f"] & \\ & & Y_A \end{tikzcd}$$

$$\partial_A f = \left(\begin{tikzcd} \partial \mathcal{A}^A \arrow[d, "i^A"] \\ \mathcal{A}^A \end{tikzcd}\right) \cdot_{\mathcal{A}\lrcorner} \left(\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}\right)$$

$X = \mathrm{sk}^X_{-1} Y$

Theorem. For any map $f \colon X \to Y$ of $[\mathcal{A}, \mathcal{M}]$ we get $Y = \operatorname{colim}_{n<\lambda} \mathrm{sk}^X_n Y$ and

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\Sigma i_A \cdot_{\mathcal{A}\lrcorner} \partial_A f"] & \mathrm{sk}^X_{<n} Y \arrow[d] \\ \bullet \arrow[r] & \mathrm{sk}^X_n Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$$\left(\begin{tikzcd} 0 \arrow[d, dashed] \\ \mathrm{sk}_n \mathcal{A}(-,-) \arrow[d, dashed] \\ \mathcal{A}(-,-) \end{tikzcd}\right) \cdot_{\mathcal{A}\lrcorner} \left(\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}\right) = \begin{tikzcd} X \arrow[d, dashed] \\ \mathrm{sk}^X_n Y = \mathrm{sk}_n Y +_{\mathrm{sk}_n X} X \arrow[d, dashed] \\ Y \end{tikzcd}$$

Theorem. For a map $f \colon X \longrightarrow Y$ of $[\mathcal{A}, \mathcal{M}]$ TFAE

- $\forall A$: $\partial_A f$ is a (trivial) cofibration
- $f \in \operatorname{cell}\{ i_A \cdot_{\lrcorner} i \mid A \in \mathcal{A},\ i \in \mathcal{M} \text{ a (trivial) cofibration} \}$

$$\begin{tikzcd} \partial\mathcal{A}_A \cdot L +_{\partial\mathcal{A}_A \cdot K} \mathcal{A}_A \cdot K \arrow[d] \\ \mathcal{A}_A \cdot L \end{tikzcd}$$

These maps are called Reedy (trivial) cofibrations.

Proof. The implication "$\Downarrow$" is the previous theorem.

For the implication "$\Uparrow$" observe that ∂_A commutes with all cellular constructions so that it is enough to study $\partial_A (i_B \cdot_{\lrcorner} i)$:

$$\left(\begin{tikzcd} \partial\mathcal{A}^A \arrow[d, "i^A"] \\ \mathcal{A}^A \end{tikzcd}\right) \cdot_{\mathcal{A}\lrcorner} \left(\begin{tikzcd} \partial\mathcal{A}_B \arrow[d, "i_B"] \\ \mathcal{A}_B \end{tikzcd} \cdot_{\lrcorner} \begin{tikzcd} K \arrow[d, "i"] \\ L \end{tikzcd} \right) = \underbrace{\left(\begin{tikzcd} \partial\mathcal{A}^A \arrow[d, "i^A"] \\ \mathcal{A}^A \end{tikzcd} \times_{\mathcal{A}\lrcorner} \begin{tikzcd} \partial\mathcal{A}_B \arrow[d, "i_B"] \\ \mathcal{A}_B \end{tikzcd} \right)}_{\text{either iso for } A \neq B} \cdot_{\lrcorner} \begin{tikzcd} K \arrow[d, "i"] \\ L \end{tikzcd}$$

$\mathcal{A}_{\neq 1}(B,A) \hookrightarrow \mathcal{A}(B,A)$ for $A = B$

$\Downarrow$

pushout of $0 \to 1$

$\underbrace{\hphantom{pushout of 0 to 1}}_{\text{pushout of } i}$

$\square$

Dually, we denote

$$\begin{tikzcd} M_A X = \{\partial\mathcal{A}_A, X\}_{\mathcal{A}} \\ X_A = \{\mathcal{A}_A, X\}_{\mathcal{A}} \arrow[u, "\delta_A X"] \arrow[u, "\{i_A, X\}_{\mathcal{A}}"'] \end{tikzcd}$$

and more generally for $f \colon X \longrightarrow Y$:

$X_A \longrightarrow M_A X$

$\delta_A f = \{i_A, f\}_{A\ulcorner}$ = pullback corner map in

$$\begin{tikzcd} \text{[illegible]} \arrow[d] & \text{[illegible]} \arrow[d] \\ Y_A \arrow[r] & M_A Y \end{tikzcd}$$

Theorem. TFAE

- $\forall A \in \mathcal{A}$: $\delta_A f$ is a (trivial) fibration
- $f \in$ cocell $\{\{i^A, p\}_\ulcorner \mid A \in \mathcal{A}, p \in \mathcal{M}$ a (trivial) fibration$\}$

$\rightsquigarrow$ Reedy (trivial) fibrations.

Theorem. There is a model structure on $[\mathcal{A}, \mathcal{M}]$, called the Reedy model structure with $\mathcal{C}$ = Reedy cofibrations, $\mathcal{F}$ = Reedy fibrations, $\mathcal{W}$ = pointwise weak equivalences.

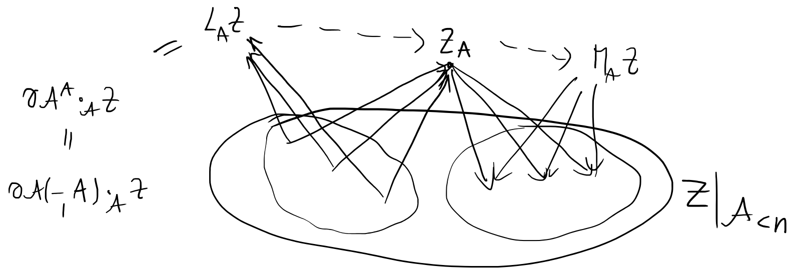
Proof. We need to show that $\mathcal{W} \cap \mathcal{C}$ are exactly the Reedy trivial cofibrations. Easily $i_A \,\lrcorner\, j \,\square\, \mathcal{F} \iff j \,\square\, \underbrace{\{i_A, \mathcal{F}\}_{A\lrcorner}}_{\text{fibrations by definition}}$.

$\overset{\uparrow}{\text{trivial cofibration}}$

inductively

$$\begin{tikzcd} L_A X \arrow[r, tail, "\sim"] \arrow[d, tail] & L_A Y \arrow[d] \arrow[dd, bend left, tail] \\ X_A \arrow[r, tail, "\sim"] \arrow[rd] & \bullet \arrow[d, tail] \\ & Y_A \end{tikzcd}$$

The factorizations are produced inductively using the following idea:



to give an extension of Z from $A_{<n}$ to $A_{\leq n}$ we need $\forall A$ of degree n to factor

$$\underbrace{L_A Z \longrightarrow Z_A \longrightarrow M_A Z}_{\text{obtained from } Z|_{A_{<n}}}$$

Now

$$\begin{tikzcd} L_A X \arrow[rr] \arrow[d] & & L_A Z \arrow[rr] \arrow[ld] & & L_A Y \arrow[d] \\ X_A \arrow[r] \arrow[d] & \bullet \arrow[r, tail] \arrow[ul, phantom, "\ulcorner"] & Z_A \arrow[r, two heads, "\sim"] & \bullet \arrow[r] \arrow[ld] \arrow[dr, phantom, "\lrcorner"] & Y_A \arrow[d] \\ M_A X \arrow[rr] & & M_A Z \arrow[rr] & & M_A Y \end{tikzcd}$$

$\square$

Application. • Properness in $\mathcal{M}_c / \mathcal{M}_f$ differently.

• Given $A \in \mathcal{M}_c$ consider $\mathrm{cst}\, A \in [\Delta, \mathcal{M}]$ and consider its Reedy cofibrant replacement $A^* \in [\Delta, \mathcal{M}]$

$$\begin{tikzcd} A^* \arrow[r] \arrow[rr, bend left] & \bar{A}^* \arrow[r] & \mathrm{cst}\, A \\ sk_0 A^* \arrow[u] \arrow[rr] & & sk_0\, \mathrm{cst}\, A \arrow[ul] \arrow[u] \arrow[ul, phantom, "\llcorner"] \end{tikzcd}$$

$\bar{A}^*$ ← does the job

$sk_0\, \mathrm{cst}\, A$ ← cofibrant when A is $(= \Delta_0 \cdot A)$

$$0 \rightarrowtail A^* \xrightarrow{\sim} \mathrm{cst}\, A$$

– we may achieve that $A^*_0 = A$ and then A^* is called a cosimplicial frame on A. Any two are related by a zig-zag of cosimplicial frames. Thus for $X \in \mathcal{M}_f$

$$\mathcal{M}(A^*, X) \in sSet$$

Dually $\mathcal{M}(A, X_*) \in sSet$ via a simplicial frame on X.

$$\left.\begin{array}{l} A^*: \Delta \to \mathcal{M} \\ K: \Delta^{op} \to Set \end{array}\right\} \text{ get } K \cdot_\Delta A^* \in \mathcal{M},$$

an action of sSet on $\mathcal{M}_c$

$K * A \overset{det}{=} K \cdot_\Delta A^*$, but not associative

The bifunctor $sSet \times c\mathcal{M} \xrightarrow{- \cdot_\Delta -} \mathcal{M}$ is close to being left Quillen

cofibrations on the left: $i^n \cdot_\Delta \mathcal{C}_R \subseteq \mathcal{C}$

$i^n \cdot_\Delta \mathcal{W} \cap \mathcal{C}_R \subseteq \mathcal{W} \cap \mathcal{C}$

trivial cofibrations on the left more subtle: $(\underbrace{\Delta^{n-1} \xrightarrow{\sim} \Delta^n}) \cdot_\Delta (X \to Y)$:

Reedy cof. between frames

$$\begin{tikzcd} X_{n-1} \arrow[r] \arrow[d, "\sim"'] & Y_{n-1} \arrow[d] \arrow[dd, bend left, "\sim"] \\ X_n \arrow[r] \arrow[rd] & \cdot \arrow[d] \arrow[ul, phantom, "\ulcorner"] \\ & Y_n \end{tikzcd}$$

these "generate" anodyne extensions in some sense

$\Delta^0 + \Delta^0 = \partial\Delta^1 \rightarrowtail \Delta^1 \xrightarrow{\sim} \Delta^0$ cylinder on Δ^0

$A + A = \underbrace{\partial\Delta^1 \cdot_\Delta A^*}_{L^1 A} \rightarrowtail \underbrace{\Delta^1 \cdot_\Delta A^*}_{A^1} \xrightarrow{\sim} A$ cylinder on A

Balancing. $\mathcal{M}(A^*, X_*) \in s^2Set = [\Delta^{op} \times \Delta^{op}, Set]$

comparison maps:

$$\begin{tikzcd} \mathcal{M}(A^*, X_*) & \mathcal{M}(cst\, A, X_*) \arrow[l] \\ \mathcal{M}(A^*, X) \arrow[u] & \end{tikzcd}$$

ptwise w.e. between Reedy cof. obj's

$\mathcal{M}(A^n, X_*) \xleftarrow{\sim} \mathcal{M}(cst A, X_*)$ since $A^n \xrightarrow{\sim} cst\, A$

Take the geometric realizations:

$$\Delta^\bullet \cdot_\Delta \mathcal{M}(A^*, X_*) \xleftarrow{\sim} \Delta^\bullet \cdot_\Delta \mathcal{M}(cst\, A, X_*) = \mathcal{M}(A, X_*)$$

$\overset{?}{\parallel}$

$\text{diag}\, \mathcal{M}(A^*, X_*)$

then this + the dual statement proves the balancing

this is about bisimplicial sets $K_{\bullet\bullet} = (K_\bullet)$.

$$\Delta^\bullet \cdot_\Delta K_\bullet = \int^{n \in \Delta} \underset{sSet}{\Delta^n} \times \underset{sSet}{K_n} = \int^{n \in \Delta} \underset{sSet}{\Delta^n} \times \left(\int^{k \in \Delta} \underset{sSet}{\Delta^k} \cdot \underset{Set}{K_{nk}} \right)$$

$$= \int^{n,k} \underset{sSet}{(\Delta^n \times \Delta^k)} \cdot \underset{Set}{K_{nk}} = \int^{(n,k)} \Delta \times \Delta (\text{diag} -, (n,k)) \cdot K_{nk}$$

$$\overset{\text{Yoneda}}{=} \text{diag}\, K$$

Application. $A = (0 \xleftarrow{-} 1 \xrightarrow{+} 2)$.

$$[A, \mathcal{M}] \underset{cst}{\overset{colim}{\rightleftarrows}} \mathcal{M}$$

cst pres w.e. (always)

and fibrations: $p: X \twoheadrightarrow Y$

$\Rightarrow$ colim takes w.e.'s between Reedy cofibrant diagrams to w.e.'s

$i_2 \cdot_A p$

$i_1 \cdot_A p$

$$\begin{tikzcd} X \arrow[rr] & & Y \\ X \arrow[u] \arrow[rr] \arrow[rd] \arrow[dd] & & Y \arrow[u] \arrow[dd, "="] \\ & \cdot \arrow[ru] \arrow[ld, "="] \arrow[rd, phantom, "\lrcorner"] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

(would not work for

$\cup$

(would not work for
$A = (0 \Leftarrow 1 \Rightarrow 0)$)

$$\begin{tikzcd}
 & {} \arrow[d] \arrow[d, shift left=3, "="] \arrow[dr, phantom, "\lrcorner"] & {} \arrow[d, "="] \\
i_1 \otimes_A P \arrow[ur, dash, bend left] & X \arrow[r] & Y \\
 & i_0 \otimes_A P \arrow[ur, bend right] &
\end{tikzcd}$$

Application. $\mathcal{M}$ simplicial

$[A^{op}, sSet] \times [A, \mathcal{M}] \xrightarrow{- \otimes_A -} \mathcal{M}$ is left Quillen

$\underset{\text{ptwise}}{\text{proj.}}$ $\qquad$ $\underset{\text{proj.}}{\text{ptwise}}$

$Q* = N(-/A)$ a particular cofibrant replacement of $*$

$$\begin{tikzcd}
\Rightarrow & Q* \otimes_A - \arrow[d] & : [A, \mathcal{M}_c] \longrightarrow \mathcal{M}_c \\
 & * \otimes_A - & \\
 & \operatorname{colim}_A \arrow[u, equal] &
\end{tikzcd}$$

preserves w.e.'s

is a w.e. on proj. cof. diagrams

$\Rightarrow N(-/A) \otimes_A - = \operatorname{hocolim}_A$

This can be extended to all model categories via frames:

$N(-/A) : \Delta^{op} \times A^{op} \longrightarrow Set$

$D^* : \Delta \times A \longrightarrow \mathcal{M}_c$ $\qquad$ frame on $D : A \to \mathcal{M}_c$

$\rightsquigarrow$ $N(-/A) \cdot_{\Delta \times A} D^* =: \operatorname{hocolim}_A D$

$\quad \shortparallel$

$N(-/A) \otimes_A D$ $\quad$ if we denote $\quad K \otimes X = K \cdot_\Delta X^*$ $\quad$ the "action" of sSet on $\mathcal{M}$

Bousfield localizations

$\mathcal{M} \rightsquigarrow Ho(\mathcal{M}) \sim \mathcal{M}_{cf}/\text{htpy}$

What should a subcategory be?

We want to limit fibrant / cofibrant objects.

(homotopy) injectivity ≐ reflective subcat's — projectivity

Example.

- $Ab \subseteq Gp$

$$\begin{tikzcd}
\mathbb{Z} * \mathbb{Z} \arrow[r] \arrow[d] & A \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[ur, dashed] &
\end{tikzcd}$$

- constant diagrams $D : \Delta \to \mathcal{M}$

$\mathcal{M} = Set$:

$$\begin{tikzcd}
\Delta_m \arrow[r] \arrow[d] & D \\
\Delta_n \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd} \equiv \begin{tikzcd}
D_m \\
D_n \arrow[u, "\cong"']
\end{tikzcd}$$

$\mathcal{M}$ general: $\forall X$

$$\begin{tikzcd}
\Delta_m \cdot X \arrow[r] \arrow[d] & D \\
\Delta_n \cdot X \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd} \equiv \begin{tikzcd}
\mathcal{M}(X, D_m) \\
\mathcal{M}(X, D_n) \arrow[u, "\cong"']
\end{tikzcd}$$

- diagrams $D : \mathcal{N} \to \mathcal{M}$ st. $D_n \xrightarrow{\cong} D_1 \times \cdots \times D_1$

$\mathcal{M} = Set$:

$$\begin{tikzcd}
\mathcal{N}_1 + \cdots + \mathcal{N}_1 \arrow[r] \arrow[d] & D \\
\mathcal{N}_n \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd} \equiv \begin{tikzcd}
D_1 \times \cdots \times D_1 \\
D_n \arrow[u, "\cong"']
\end{tikzcd}$$

- diagrams $D : \mathcal{N} \to \mathcal{M}$ st. $D_n \xrightarrow{\cong} \{K, D_{n+1}\}$

$$\begin{tikzcd}
\mathcal{N}_{n+1} \otimes K \arrow[r] \arrow[d] & D \\
\mathcal{N}_n \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd} \qquad \begin{tikzcd}
\{K, D_{n+1}\} \\
D_n \arrow[u, "\cong"']
\end{tikzcd}$$

$\mathcal{V} = Top_* / sSet_*$

$K = S^1$

$\rightsquigarrow D_n \to \Omega D_{n+1}$

- sheaves

$$\begin{tikzcd}
\operatorname{colim} \mathcal{X}_{U_i} \arrow[r] \arrow[d] & D \\
\mathcal{X}_U \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd} \qquad \begin{tikzcd}
\lim D_{U_i} \\
D_U \arrow[u, "\cong"']
\end{tikzcd}$$

$\mathcal{X} = Op(X)$

U_i open covering of U closed under intersections

Example. where htpy version is necessary

$\mathcal{M} = Top$:

$$\begin{tikzcd}
S^n \arrow[r] \arrow[d] & X \\
D^{n+1} \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

∃ unique up to higher htpies

$\pi_n X = 0$

?

$$\begin{tikzcd}
map(S^n, X) \\
map(D^{n+1}, X) \arrow[u, "\sim"']
\end{tikzcd}$$

$\pi_n X = 0$

$\pi_{n+1} X = 0$

$\vdots$

more generally $A \to X$ $\quad map(A, X)$

more generally

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \\ B \arrow[ur, dashed, "\exists!_h"'] & \end{tikzcd} \equiv \begin{tikzcd} \mathrm{map}(A,X) \\ \mathrm{map}(B,X) \arrow[u, two heads, "\sim"'] \end{tikzcd}$$

better:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \\ B \arrow[ur, dashed, "\exists!_h"'] & \end{tikzcd} \equiv \begin{tikzcd} \mathrm{map}(A,X) & L \arrow[l] \\ \mathrm{map}(B,X) \arrow[u, two heads, "\sim"'] & K \arrow[l] \arrow[u, tail] \arrow[ul, phantom] \end{tikzcd} \begin{tikzcd} & \\ \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{map}(A,X) & L \arrow[l] \arrow[dl, dashed] \\ \mathrm{map}(B,X) \arrow[u, two heads, "\sim"'] & K \arrow[l] \arrow[u, tail] \end{tikzcd} \equiv \begin{tikzcd} K\times B +_{K\times A} L\times A \arrow[r] \arrow[d] & X \\ L\times B \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Will assume $\mathcal{M}$ left proper, cellular = cofibrantly generated with cofibrations effective mono's

Definition. Let $f: A \to B$ be a cofibration between cofibrant objects. We say that W is *f-local* if it is fibrant and

$$f^*: \mathrm{map}(B,W) \xrightarrow{\sim} \mathrm{map}(A,W)$$

is a weak equivalence of simplicial sets. $\Big\} \left(\mathcal{J} \cup \{L^nB +_{L^nA} A^n \to B^n\}\right)^{\boxslash}$

Definition. A map $g: X \to Y$ is an *f-local equivalence* if its cofibrant replacement $\tilde{g}: \tilde{X} \rightarrowtail \tilde{Y}$

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, equal] & \tilde{X} \arrow[r, "\sim"] \arrow[d, tail, "\tilde{g}"] & X \arrow[d, "g"] \\ 0 \arrow[r, tail] & \tilde{Y} \arrow[r, "\sim"] & Y \end{tikzcd}$$

gives, for each f-local W, a w.e. $\tilde{g}^*: \mathrm{map}(\tilde{Y},W) \xrightarrow{\sim} \mathrm{map}(\tilde{X},W)$

(any two related by a zig-zag of w.e.'s of such $\Rightarrow$ independent of choice)

Definition. An *f-localization* of X is an f-local equivalence

$j: X \longrightarrow \hat{X}$ with $\hat{X}$ f-local.

Aim. Construct an "f-local" model structure in which:

- f-local = fibrant ; fibrations are complicated BUT cofibrations of $\mathcal{M}$
- f-local equivalence = weak equivalence
- f-localization = fibrant replacement = "reflection"

$\to$ better: $\mathrm{Id}: \mathcal{M} \rightleftarrows \mathcal{M}^{f\text{-local}} : \mathrm{Id}$

preserves
cof. & w.e.

is interpreted as $\mathcal{M}_{cf} \underset{f\text{-localization}}{\rightleftarrows} \mathcal{M}^{f\text{-local}}_{cf}$