Reminder on weighted colimits

$X \times_G Y$

$(xg,y)=(x,gy)$

$$\begin{tikzcd} X\times G\times Y \arrow[d, shift right, "\mathrm{act}_X\times 1"'] \arrow[d, shift left, "1\times \mathrm{act}_Y"] \\ X\times Y \end{tikzcd}$$

$X : G^{op} \to Set$

$Y : G \to Set$

$M \otimes_R N$

$xr\otimes y = x\otimes ry$

$$\begin{tikzcd} M\otimes R\otimes N \arrow[d, shift right] \arrow[d, shift left] \\ M\otimes N \end{tikzcd}$$

$M : R^{op} \to Ab$

$N : R \to Ab$

$W \otimes_{\mathcal{A}} D$

$$\begin{tikzcd} \sum_{b,c} Wc\otimes \mathcal{A}(b,c)\otimes Db \arrow[d, shift right] \arrow[d, shift left] \\ \sum_a Wa\otimes Da \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A}(b,c)\otimes Db \arrow[d] \\ Dc \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}(b,c) \arrow[d, "D"] \\ \mathcal{M}(Db,Dc) \end{tikzcd}$$

$W : \mathcal{A}^{op} \to \mathcal{V}$

$D : \mathcal{A} \to \mathcal{M}$

$\Delta^\bullet \times_\Delta X$

$(\theta^* t, x) \sim (t, \theta_* x)$

$$\begin{tikzcd} \Delta^n & - - - - \arrow[d, shift right] \arrow[d, shift left] & X_n \arrow[d, "d_i"] \\ \Delta^{n-1} \arrow[u, "d^i"] & \sum \Delta^n \times X_n & X_{n-1} \end{tikzcd}$$

$\Delta^\bullet : \Delta \to sSet$

$X : \Delta^{op} \to Top$ ....

$W\otimes_{\mathcal{A}} D = \int^{A\in\mathcal{A}} WA\otimes DA$

$\otimes : \mathcal{M}\times\mathcal{N} \to \mathcal{P}$

$\otimes_{\mathcal{A}} : [\mathcal{A}^{op},\mathcal{M}]\times[\mathcal{A},\mathcal{N}] \to \mathcal{P}$

$$\sum_{b,c} Wc\otimes \mathcal{A}(b,c)\otimes Db \rightrightarrows \sum_a Wa\otimes Da \xrightarrow[\text{coend}]{} W\otimes_{\mathcal{A}} D \qquad / \ \mathcal{M}(-,N)$$

$$\prod_{b,c}\{\mathcal{A}(b,c), \mathcal{M}(Wc\otimes Db, N)\} \leftleftarrows \prod_a \mathcal{M}(Wa\otimes Da, N) \leftarrow \mathcal{M}(W\otimes_{\mathcal{A}}D, N)$$

$$\| \qquad\qquad \|$$

$$\prod_{b,c}\{\mathcal{A}(b,c), \mathcal{V}(Wc, \mathcal{M}(Db,N))\} \leftleftarrows \prod_a \mathcal{V}(Wa, \mathcal{M}(Da,N)) \leftarrow [\mathcal{A}^{op},\mathcal{V}](W, \mathcal{M}(D,N))$$

end W-weighted cones $D \rightrightarrows N$

$\mathcal{M}(DA,N) \overset{\text{Yoneda}}{=} [\mathcal{A}^{op},\mathcal{V}](\mathcal{A}(-,A), \mathcal{M}(D,N)) \overset{\text{def}}{\cong} \mathcal{M}(\mathcal{A}(-,A)\otimes_{\mathcal{A}} D, N)$

Example.

$\mathcal{A}(-,A)\otimes_{\mathcal{A}} D = DA$ dually $\{\mathcal{A}(A,-), D\}_{\mathcal{A}} = DA$

$\mathcal{A}(-,-)\otimes_{\mathcal{A}} D = D \leftarrow R\otimes_R M = M \qquad \mathrm{Hom}_R(R,M) = M$

$\mathcal{A}^{op} \to [\mathcal{B},\mathcal{V}]$

$W : \mathcal{A}^{op}\times\mathcal{B} \to \mathcal{V}$

$D : \mathcal{A} \to \mathcal{M}$

$\Big\} \quad W\otimes_{\mathcal{A}} D : \mathcal{B} \to \mathcal{M}$

${}_S M_R \otimes_R {}_R N$

Restriction of scalars : $F : \mathcal{B} \to \mathcal{A}$

$\mathcal{A}(-,F-) : \mathcal{A}^{op}\times\mathcal{B} \to \mathcal{V}$ $\qquad D : \mathcal{A} \to \mathcal{M}$

$\mathcal{A}(-,F-)\otimes_{\mathcal{A}} D = DF$ $\qquad$ dually $\{\mathcal{A}(F-,-), D\}_{\mathcal{A}} = DF$

R $\qquad R \to S$ $\qquad {}_{\mathbb{R}}\mathbb{C}_{\mathbb{C}} \otimes_{\mathbb{C}} V = V^{\mathbb{R}}$ $\qquad \overset{\|}{F^*D}$

${}_R S_S \otimes_S {}_S M = {}_R M$

Adjunction :

$$[\mathcal{B},\mathcal{V}](C, \overbrace{\{\mathcal{A}(F-,-), D\}_{\mathcal{A}}}^{F^*D}) = [\mathcal{A},\mathcal{V}](\underbrace{\mathcal{A}(F-,-)\otimes_{\mathcal{B}} C}, D)$$

$\mathrm{Hom}_R(\underset{R}{M}, \underset{S}{\mathrm{Hom}_S(S,N)}) = \mathrm{Hom}_S(S\otimes_R M, N)$ $\qquad F_!C$ extension of scalars (like $S\otimes_R L$)

$[\mathcal{B},\mathcal{V}](C, F^*D) = [\mathcal{A},\mathcal{V}](F_!C, D)$

$\uparrow \ \mathrm{Lan}_F C$ $\qquad$ left Kan extension

eg. $F : \mathcal{B} \hookrightarrow \mathcal{A}$ full embedding fully faithful

$$\begin{tikzcd} \mathcal{B} \arrow[r, "C"] \arrow[d, hook, "F"'] & \mathcal{M} \\ \mathcal{A} \arrow[ur, "\mathrm{Lan}_F C"', "/\!/"] & \end{tikzcd}$$

$$F_! C\,(B) = \mathcal{A}(F-, B) \otimes_{\mathcal{B}} C$$
$$= \mathcal{B}(-, B) \otimes_{\mathcal{B}} C$$
$$= C_B \quad \text{real extension}$$

$$\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{D} \mathcal{M}$$

$$\mathcal{A} \xrightarrow{D} \mathcal{M} \xrightarrow{1} \mathcal{M}$$

$$W \otimes_{\mathcal{A}} D = \ldots \otimes_{\mathcal{M}} 1$$

$$W \otimes_{\mathcal{B}} (D \circ F) = \overline{W} \otimes_{\mathcal{A}} D$$

$$W \otimes_{\mathcal{B}} \underbrace{\left(\mathcal{A}(-, F-) \otimes_{\mathcal{A}} D\right)}_{D \circ F} = \underbrace{\left(W \otimes_{\mathcal{B}} \mathcal{A}(-, F-)\right)}_{\overline{W}} \otimes_{\mathcal{A}} D$$

$$\overline{W} = \mathrm{Lan}_F W$$

restricted from S to R

$$M \otimes_R N = M \otimes_R (S \otimes_S N)$$
$$= (M \otimes_R S) \otimes_S N$$

---

weighted colimit

$$\mathrm{Hom}(W \otimes_{\mathcal{A}} D, X) \cong \mathrm{Hom}_{\mathcal{A}}(W, \mathrm{Hom}(D, X))$$

$$\uparrow$$
$$[\mathcal{A}^{op}, \mathcal{V}]$$

$\mathcal{M}$

$$\mathcal{A}(-, A)$$
$$/\!/$$

- $\mathcal{A}^A \otimes_{\mathcal{A}} D \cong D_A$ Yoneda

  $(R \otimes_R M \cong M)$

- $(\mathrm{colim}\, W_i) \otimes_{\mathcal{A}} D \cong \mathrm{colim}\,(W_i \otimes_{\mathcal{A}} D)$

- $K \otimes_* X = K \otimes X$

Reedy model categories, framings

$\mathcal{M}(M,N)$

Motivation. Any model category $\mathcal{M}$ is in some weak sense enriched over sSet and, as a result, $Ho(\mathcal{M})$ will be enriched over $Ho(sSet)$.

Start with an honest sSet-model category = simplicial model category

- since $K \in sSet$ is a colimit $K = \operatorname{colim}_{(n,\Delta^n \to K)} \Delta^n = K \cdot_\Delta \Delta^\bullet$

$\Delta^\bullet : \Delta \to sSet$
$K : \Delta^{op} \to Set$

we have $K \otimes M = \operatorname{colim}(\Delta^n \otimes M)$ and it is enough to give $\Delta^n \otimes M$

- clearly $\Delta^0 \otimes M \cong M$, since Δ^0 is the monoidal unit
- what is the essential property of $\Delta^1 \otimes M$?

  it is a cylinder ... at least for $M \in \mathcal{M}_c$:

$$\Delta^0 + \Delta^0 \overset{[d^0,d^1]}{\rightarrowtail} \Delta^1 \xrightarrow[\sim]{s^0} \Delta^0 \quad \rightsquigarrow \quad M + M \rightarrowtail \Delta^1 \otimes M \xrightarrow{\sim} M$$

- what about $\Delta^2 \otimes M$?

$$\begin{tikzcd}
 & & \Delta^0 \arrow[dl] \arrow[dr, "d^0"] & & \\
 & \Delta^1 \arrow[r, "d^1"] & \Delta^2 & \Delta^1 \arrow[l, "d^0"'] & \\
\Delta^0 \arrow[ur] \arrow[rr, "d^1"'] & & \Delta^1 \arrow[u, "d^2"] & & \Delta^0 \arrow[ll, "d^0"] \arrow[ul, "d^1"']
\end{tikzcd}$$

, more compactly $\partial\Delta^2 \to \Delta^2$, $\partial\Delta^2 = L_2\Delta^\bullet$

$$\rightsquigarrow \quad L_2\Delta^\bullet \otimes M \rightarrowtail \Delta^2 \otimes M \xrightarrow{\sim} \overset{\mathcal{M}}{\cancel{M_2\Delta^\bullet}}\otimes M$$

$$\begin{tikzcd}
 & \Delta^2 \arrow[dl, "s^0"'] \arrow[dr, "s^1"] & \\
\Delta^1 \arrow[dr, "s^0"'] & & \Delta^1 \arrow[dl, "s^0"] \\
 & \Delta^0 &
\end{tikzcd}$$

, more compactly $\Delta^2 \xrightarrow{\sim} M_2\Delta^\bullet$ $\quad$ $\forall$ $\Delta^\bullet : \Delta \to sSet$ a cosimplicial object in sSet

- need some calculus of such diagrams $\Delta \to sSet$, $\Delta \to \mathcal{M}$ $\quad \rightsquigarrow \Delta^\bullet \otimes M : \Delta \to \mathcal{M}$ a cosimplicial object in $\mathcal{M}$

  → Reedy categories, Reedy model structures

A Reedy category has two kinds of maps – direct and inverse (like d^i and s^i in Δ)

Definition A direct category is a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A} \to \lambda$ (ordinal) that satisfies $f : A \to B \rightsquigarrow f = 1 \Leftrightarrow \deg A = \deg B$

$\Rightarrow \deg A \leq \deg B$ $\quad$ $f \neq 1 \Rightarrow \deg A < \deg B$

An inverse category is a dual notion, i.e. a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A}^{op} \to \lambda$ satisfying $f = 1 \Leftrightarrow \deg A = \deg B$.

A Reedy category is a category $\mathcal{A}$ together with two subcategories $\mathcal{A}^+$, $\mathcal{A}_-$ and a function $\deg : \operatorname{ob}\mathcal{A} \to \lambda$ that

- makes $\mathcal{A}^+$ into a direct category
- makes $\mathcal{A}_-$ into an inverse category
- any morphism has a unique decomposition:

$$\sum_{A \in \mathcal{A}} \mathcal{A}^+(A,C) \times \mathcal{A}_-(B,A) \xrightarrow[\cong]{\circ} \mathcal{A}(B,C)$$

$$\begin{tikzcd}
\bullet \arrow[dr, "\mathcal{A}_-"'] \arrow[rr] & & \bullet \\
 & A \arrow[ur, "\mathcal{A}_+"'] &
\end{tikzcd}$$

- any morphism

$\mathcal{A}$

$$\sum_{A\in\mathcal{A}} \mathcal{A}^+(A,C)\times\mathcal{A}_-(B,A) \xrightarrow[\cong]{\circ} \mathcal{A}(B,C)$$

as a composition of an inverse and a direct morphism

The Yoneda lemma gives $X \cong \mathcal{A}(-,-)\cdot_{\mathcal{A}} X$ and we will describe a way of building X by decomposing $\mathcal{A}(-,-)$. The axioms actually give

$$\mathcal{A}(-,-) = \sum_{A\in\mathcal{A}} \mathcal{A}^+(A,-)\times\mathcal{A}_-(-,A) = \sum_{A\in\mathcal{A}} \mathcal{A}^+_A \times \mathcal{A}^A_-$$

(but only of functors $\mathcal{A}^{op}_-\times\mathcal{A}^+\to Set$ ⚠)

but in order to understand $\mathcal{A}(-,-)$ we factor it into

$sk_n\,\mathcal{A}(-,-)$ ... maps that factor through some A of degree $\leq n$

and clearly we have

$$\mathcal{A}(-,-) = \operatorname*{colim}_{n<\lambda} sk_n\,\mathcal{A}(-,-)$$

so that it remains to study the difference between $sk_n\mathcal{A}$ and $sk_{n-1}\mathcal{A}$ or, better for n limit, $sk_{<n}\mathcal{A} = \operatorname*{colim}_{i<n} sk_i\,\mathcal{A}$.

Quite clearly, we have a pushout square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & sk_{<n}\mathcal{A} \arrow[d] \\ \sum\limits_{A\in\mathcal{A}_n} \mathcal{A}^+_A\times\mathcal{A}^A_- \arrow[r] & sk_n\,\mathcal{A} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

with A ranging over all objects of degree n

However, it will be crucial to express this in terms of representable functors on $\mathcal{A}$, rather than on $\mathcal{A}^+$, $\mathcal{A}_-$.

Examples.

- any ordinal is a direct category
- $\begin{tikzcd}[row sep=small] & \bullet \\ \bullet \arrow[ur, "+"] \arrow[dr, "+"'] & \\ & \bullet \end{tikzcd}$ is a direct category
- $\bullet \xleftarrow{-} \bullet \xrightarrow{+} \bullet$ is a Reedy category
- Δ^+ is a direct category (non-empty finite ordinals + monos)
- Δ_- is an inverse category (non-empty finite ordinals + epis)
- Δ is a Reedy category

$\mathcal{A}$ direct $\Longleftrightarrow$ $\mathcal{A}^{op}$ inverse

$\mathcal{A}$ Reedy $\Longleftrightarrow$ $\mathcal{A}^{op}$ Reedy

$\Longrightarrow$ Δ^{op} Reedy

Notation. We denote

- $\mathcal{A}_A = \mathcal{A}(A,-) \in [\mathcal{A}, Set]$ the covariant representable
- $\mathcal{A}^A = \mathcal{A}(-,A) \in [\mathcal{A}^{op}, Set]$ the contravariant representable ... think Δ^n

We further define two subfunctors

- $i_A: \partial\mathcal{A}_A \subseteq \mathcal{A}_A$ of maps that factor through an object of lower degree.
- $i^A: \partial\mathcal{A}^A \subseteq \mathcal{A}^A$ of maps that factor through an object of lower degree.

In the decomposition

$$\mathcal{A}_A = \mathcal{A}(A,-) = \sum \mathcal{A}^+(B,-)\times\mathcal{A}_-(A,B)$$

$$\partial\mathcal{A}_A = \sum \mathcal{A}^+(B,-)\times\partial\mathcal{A}_-(A,B)$$

↖ only 1 is excluded

(coproduct)

This means that there is a pushout (coproduct)

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \partial\mathcal{A}_A \arrow[d, "i_A"] \arrow[dl, phantom, "\ulcorner", very near end] \\ \mathcal{A}^+_A \arrow[r] & \mathcal{A}_A \end{tikzcd}$$

Dually, we get

$$\mathcal{A}^A = \mathcal{A}(-,A) = \sum \mathcal{A}^+(B,A) \times \mathcal{A}_-(-,B)$$

$$\partial\mathcal{A}^A = \sum \partial\mathcal{A}^+(B,A) \times \mathcal{A}_-(-,B)$$

and a pushout (coproduct)

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \partial\mathcal{A}^A \arrow[d, "i^A"] \arrow[dl, phantom, "\ulcorner", very near end] \\ \mathcal{A}^A_- \arrow[r] & \mathcal{A}^A \end{tikzcd}$$

$$= \sum \underbrace{\partial(\mathcal{A}^+)^A \times_{\mathcal{A}^+} \mathcal{A}^+(B,-)}_{\partial\mathcal{A}^+(B,A) \text{ by Yoneda}} \times \mathcal{A}_-(-,B)$$

$$= \partial(\mathcal{A}^+)^A \times_{\mathcal{A}^+} \underbrace{\sum \mathcal{A}^+(B,-) \times \mathcal{A}_-(-,B)}_{\mathcal{A}(-,-)}$$

$$= \partial(\mathcal{A}^+)^A \cdot_{\mathcal{A}^+} \mathcal{A}(-,-)$$

= extension of $\partial(\mathcal{A}^+)^A$

$$\Rightarrow \partial\mathcal{A}^A \cdot_{\mathcal{A}} X = (\partial(\mathcal{A}^+)^A \times_{\mathcal{A}^+} \mathcal{A}(-,-)) \cdot_{\mathcal{A}} X$$

$$= \partial(\mathcal{A}^+)^A \cdot_{\mathcal{A}_+} X$$

Putting these together yields that

$$\begin{tikzcd} & 0 \times 0 \arrow[dl] \arrow[dr] & \\ 0 \times \mathcal{A}^A_- \arrow[r] \arrow[dr] & \bullet \arrow[d] & \mathcal{A}^+_A \times 0 \arrow[l] \arrow[dl] \\ & \mathcal{A}^+_A \times \mathcal{A}^A_- & \end{tikzcd}$$

zero

each square pushout $\Longrightarrow$ gives as a pushout

$$\begin{tikzcd} & \partial\mathcal{A}_A \times \partial\mathcal{A}^A \arrow[dl] \arrow[dr] & \\ \partial\mathcal{A}_A \times \mathcal{A}^A \arrow[r] \arrow[dr] & \bullet \arrow[d, "i_A \times_{\lrcorner} i^A"] & \mathcal{A}_A \times \partial\mathcal{A}^A \arrow[l] \arrow[dl] \\ & \mathcal{A}_A \times \mathcal{A}^A & \end{tikzcd}$$

Now we get a diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \sum_A \partial\mathcal{A}_A \times \mathcal{A}^A \underset{\partial\mathcal{A}_A \times \partial\mathcal{A}^A}{+} \mathcal{A}_A \times \partial\mathcal{A}^A \arrow[r] \arrow[d, "\sum_A i_A \times_{\lrcorner} i^A"] \arrow[dl, phantom, "\ulcorner", very near end] & \mathrm{sk}_{<n}\,\mathcal{A}(-,-) \arrow[d] \\ \sum_A \mathcal{A}^+_A \times \mathcal{A}^A_- \arrow[r] & \sum_A \mathcal{A}_A \times \mathcal{A}^A \arrow[r] & \mathrm{sk}_n\,\mathcal{A}(-,-) \end{tikzcd}$$

sums range over $A \in \mathcal{A}$ with $\deg A = n$

in which the outer square is a pushout $\Rightarrow$ so is the one on the right but now in $[\mathcal{A}^{op} \times \mathcal{A}, \mathrm{Set}]$?

Upon applying $- \cdot_{\mathcal{A}} X$, we denote:

$$\begin{tikzcd} \partial\mathcal{A}^A \cdot_{\mathcal{A}} X \arrow[r, "i^A \cdot_{\mathcal{A}} X"] \arrow[d, equal, "\mathrm{def}"] & \overset{\mathcal{A}(-,A)}{\mathcal{A}^A} \cdot_{\mathcal{A}} X \arrow[d, equal] \\ L_A X \arrow[r, "\partial_A X"] & X_A \end{tikzcd}$$

latching object

Example. $\mathcal{A} = \Delta^{op}$, $\mathcal{M} = \mathrm{Set}$

$\Rightarrow L_n X \subseteq X_n$ the subset of deg. simplices (needs a bit of work, see above).

$0 = \mathrm{sk}_{-1} X$

$X = \mathrm{colim}\ \mathrm{sk}_n X$ and

Theorem. For any $X \in [\mathcal{A}, \mathcal{M}]$ we get

$$\begin{tikzcd} \sum_A \partial\mathcal{A}_A \cdot X_A \underset{\partial\mathcal{A}_A \cdot L_A X}{+} \mathcal{A}_A \cdot L_A X \arrow[r] \arrow[d, "i_A \cdot_{\lrcorner} \partial_A X"] & \mathrm{sk}_{<n} X \arrow[d] \arrow[dl, phantom, "\ulcorner", very near end] \\ \sum \mathcal{A}_A \cdot X_A \arrow[r] & \mathrm{sk}_n X \end{tikzcd}$$

$$\mathcal{A} = 0 \to 1 \to 2 \to \cdots$$

$$\begin{array}{ccccc} X_0 \to & X_0 & \to & X_0 & \to \cdots \\ & \cap\!\shortmid & & \cap\!\shortmid & \\ & X_1 & \to & X_1 & \to \cdots \end{array}$$

Important special case.

When $\mathcal{A}$ is direct, we have $\partial\mathcal{A}_A = 0$ and, consequently,

$$\begin{tikzcd} \sum_A \mathcal{A}_A \cdot L_A X \arrow[r] \arrow[d] & \mathrm{sk}_{<n} X \arrow[d] \arrow[dl, phantom, "\ulcorner", very near end] \\ {} & {} \end{tikzcd}$$

proj. model structure generated by

$$\begin{tikzcd}
\sum\limits_A \mathcal{A}_A \cdot L_A X \arrow[r] \arrow[d] & \mathrm{sk}_{<n} X \arrow[d] \\
\sum\limits_A \mathcal{A}_A \cdot X_A \arrow[r] & \mathrm{sk}_n X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

proj. model structure generated by

$\mathcal{A}_A \cdot K \longrightarrow \mathcal{A}_A \cdot L$

$A \in \mathcal{A}$, $K \rightarrowtail L$

so that : $\forall A \in \mathcal{A}$: $\partial_A X : L_A X \to X_A$ cofibration

next theorem

$\Longleftarrow$ X is cofibrant in the projective model structure

E.g. $X = X_0 \to X_1 \to X_2 \to \cdots$

$L_0 X \to X_0$: $0 \to X_0$

$L_n X \to X_n$: $X_{n-1} \to X_n$

More generally, for $f : X \to Y$ we denote $\partial_A f = i^A \cdot_{\mathcal{A}\lrcorner} f$, i.e. the pushout corner map in

$$\begin{tikzcd}
L_A X \arrow[r] \arrow[d, "\partial_A X"'] & L_A Y \arrow[d] \arrow[dd, bend left, "\partial_A Y"] \\
X_A \arrow[r] \arrow[dr, bend right] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[d, "\partial_A f"] \\
& Y_A
\end{tikzcd}$$

$$\partial_A f = \begin{array}{c} \partial\mathcal{A}^A \\ \downarrow{\scriptstyle i^A} \\ \mathcal{A}^A \end{array} \cdot_{\mathcal{A}\lrcorner} \begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ Y \end{array}$$

Theorem. For any map $f : X \to Y$ of $[\mathcal{A}, \mathcal{M}]$ we get

$X = \mathrm{sk}^X_{-1} Y$

$Y = \operatorname{colim}\limits_{n<\lambda} \mathrm{sk}^X_n Y$ and

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\sum i_A \cdot_{\mathcal{A}\lrcorner} \partial_A f"'] & \mathrm{sk}^X_{<n} Y \arrow[d] \\
\cdot \arrow[r] & \mathrm{sk}^X_n Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$$\begin{array}{c} 0 \\ \vdots \\ \mathrm{sk}_n \mathcal{A}(-,-) \\ \vdots \\ \mathcal{A}(-,-) \end{array} \cdot_{\mathcal{A}\lrcorner} \begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ Y \end{array} = \begin{array}{c} X \\ \vdots \\ \mathrm{sk}^X_n Y = \mathrm{sk}_n Y +_{\mathrm{sk}_n X} X \\ \vdots \\ Y \end{array}$$

Theorem. For a map $f : X \to Y$ of $[\mathcal{A}, \mathcal{M}]$ TFAE

- $\forall A$: $\partial_A f \overset{=}{} i^A \cdot_{\mathcal{A}\lrcorner} f$ is a (trivial) cofibration
- $f \in \mathrm{cell}\{ i_A \cdot_\lrcorner i \mid A \in \mathcal{A},\ i \in \mathcal{M} \text{ a (trivial) cofibration} \}$

$$\begin{tikzcd}
\partial\mathcal{A}_A \cdot L +_{\partial\mathcal{A}_A \cdot K} \mathcal{A}_A \cdot K \arrow[d] \\
\mathcal{A}_A \cdot L
\end{tikzcd}$$

These maps are called Reedy (trivial) cofibrations.

Proof. The implication "$\Downarrow$" is the previous theorem.
For the implication "$\Uparrow$" observe that ∂_A commutes with all cellular constructions so that it is enough to study $\partial_A(i_B \cdot_\lrcorner i)$:

$$\begin{array}{c} \partial\mathcal{A}^A \\ \downarrow{\scriptstyle i^A} \\ \mathcal{A}^A \end{array} \cdot_{\mathcal{A}\lrcorner} \left(\begin{array}{c} \partial\mathcal{A}_B \\ \downarrow{\scriptstyle i_B} \\ \mathcal{A}_B \end{array} \cdot_\lrcorner \begin{array}{c} K \\ \downarrow{\scriptstyle i} \\ L \end{array} \right) = \underbrace{\left(\begin{array}{c} \partial\mathcal{A}^A \\ \downarrow{\scriptstyle i^A} \\ \mathcal{A}^A \end{array} \times_{\mathcal{A}\lrcorner} \begin{array}{c} \partial\mathcal{A}_B \\ \downarrow{\scriptstyle i_B} \\ \mathcal{A}_B \end{array} \right)}_{\text{either iso for } A \neq B} \cdot_\lrcorner \begin{array}{c} K \\ \downarrow{\scriptstyle i} \\ L \end{array}$$

$\mathcal{A}_{\neq 1}(B, A) \hookrightarrow \mathcal{A}(B, A)$ for $A = B$

$\Downarrow$

pushout of $0 \to 1$

$\underbrace{}_{\text{pushout of } i}$

$$\begin{array}{c} \partial A \\ \downarrow \\ A \end{array} \otimes_\lrcorner \begin{array}{c} \partial B \\ \downarrow \\ B \end{array} = \begin{array}{c} \partial A \otimes B +_{\partial A \otimes \partial B} A \otimes \partial B \\ \downarrow \\ A \otimes B \end{array}$$

∎

Dually, we denote

$$\begin{tikzcd}
M_A X = \{\partial\mathcal{A}_A, X\}_{\mathcal{A}} \\
X_A = \{\mathcal{A}_A, X\}_{\mathcal{A}} \arrow[u, "\delta_A X", "\{i_A, X\}_{\mathcal{A}}"']
\end{tikzcd}$$

and more generally for $f : X \to Y$:

$\delta_A f = \{i_A, f\}_{A\ulcorner}$ = pullback corner map in

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & M_A X \arrow[d] \\ Y_A \arrow[r] & M_A Y \end{tikzcd}$$

Theorem. TFAE

- $\forall A \in \mathcal{A}$: $\delta_A f$ is a (trivial) fibration
- $f \in$ cocell $\{ \{i^A, p\}_\ulcorner \mid A \in \mathcal{A}, p \in \mathcal{M}$ a (trivial) fibration$\}$

$\leadsto$ Reedy (trivial) fibrations.

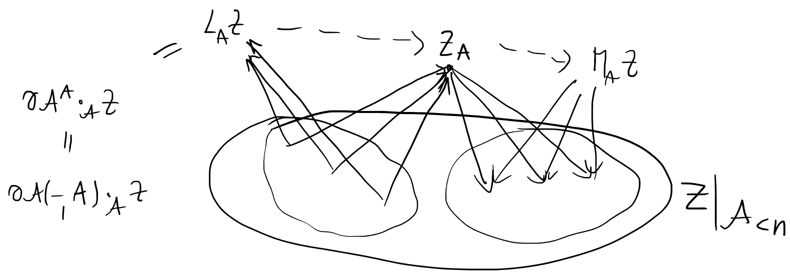
Theorem. There is a model structure on $[\mathcal{A}, \mathcal{M}]$, called the Reedy model structure with $\mathcal{C}$ = Reedy cofibrations, $\mathcal{W}$ = pointwise weak equivalences. $\mathcal{F}$ = Reedy fibrations,

Proof. We need to show that $\mathcal{W} \cap \mathcal{C}$ are exactly the Reedy trivial cofibrations. Easily $\underset{\text{trivial cofibration}}{\underset{\uparrow}{i_A \lrcorner j}} \,\square\, \mathcal{F} \iff j \,\square\, \underbrace{\{i_A, \mathcal{F}\}_{A\lrcorner}}_{\text{fibrations}}$. by definition

inductively

$$\begin{tikzcd} L_A X \arrow[r, tail, "\sim"] \arrow[d] & L_A Y \arrow[d] \arrow[dr] & \\ X_A \arrow[r, tail, "\sim"] \arrow[rr, bend right] & \bullet \arrow[r, two heads] & Y_A \end{tikzcd}$$

The factorizations are produced inductively using the following idea:



to give an extension of Z from $\mathcal{A}_{<n}$ to $\mathcal{A}_{\leq n}$ we need $\forall A$ of degree n to factor

$$\begin{tikzcd} L_A Z \arrow[r] \arrow[rr, bend right, "\text{obtained from } Z|_{A_{<n}}"'] & Z_A \arrow[r] & M_A Z \end{tikzcd}$$

Now

$$\begin{tikzcd} L_A X \arrow[r] \arrow[d] & & L_A Z \arrow[r] \arrow[dl] & & L_A Y \arrow[d] \\ X_A \arrow[r] & \bullet \arrow[r, tail] \arrow[ul, phantom, "\ulcorner"] & Z_A \arrow[r, two heads, "\sim"] & \bullet \arrow[r] \arrow[dl] & Y_A \arrow[d] \\ M_A X \arrow[rr] \arrow[u, leftarrow] & & M_A Z \arrow[rr] \arrow[ur, phantom, "\lrcorner"] & & M_A Y \end{tikzcd}$$

$\square$

Application. • Properness in $\mathcal{M}_c / \mathcal{M}_f$ differently.

• Given $A \in \mathcal{M}_c$ consider $\mathrm{cst}\, A \in [\Delta, \mathcal{M}]$ and consider its Reedy cofibrant replacement $A^* \in [\Delta, \mathcal{M}]$

$$0 \rightarrowtail A^* \overset{\sim}{\twoheadrightarrow} \mathrm{cst}\, A$$

$\mathcal{M}$ simplicial
$A^* = \Delta^* \otimes A$
$X_* = \{\Delta^*, X\} = X^{\Delta^*}$

$$\begin{tikzcd} A^* \arrow[rr] \arrow[dr] & & \mathrm{cst}\, A \\ & \bar{A}^* \arrow[ur] & \\ \mathrm{sk}_0 A^* \arrow[uu] \arrow[rr] & & \mathrm{sk}_0\, \mathrm{cst}\, A \arrow[ul] \arrow[uu] \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

does the job

cofibrant when A is $(= \Delta_0 \cdot A)$

– we may achieve that $A^0 = A$ and then A^* is called a cosimplicial frame on A. Any two are related by a zig-zag of cosimplicial frames. Thus for $X \in \mathcal{M}_f$

$$\mathcal{M}(A^*, X) \in sSet$$

$A^* : \Delta \to \mathcal{M}$, $X \in \mathcal{M}$ $\Big\}$ $\mathcal{M}(A^*, X) : \Delta^{op} \to Set$

Dually $\mathcal{M}(A, X_*) \in sSet$ via a simplicial frame on X.

$\mathcal{M}(A^*, X)_0 = \mathcal{M}(A, X)$

$\mathcal{M}(A^*, X)_1 = \mathcal{M}(Cyl A, X)$

$\Delta^n \cdot_\Delta A^* = A^n$

$$\left.\begin{array}{l} A^* : \Delta \to \mathcal{M} \\ K : \Delta^{op} \to Set \end{array}\right\} \text{ get } K \cdot_\Delta A^* \in \mathcal{M},$$

an action of $sSet$ on $\mathcal{M}_c$

$K * A \overset{def}{=} K \cdot_\Delta A^*$, but not associative

The bifunctor $sSet \times c\mathcal{M} \xrightarrow{- \cdot_\Delta -} \mathcal{M}$ is close to being left Quillen

action of $Ho(sSet)$ on $Ho(\mathcal{M}_c)$ is assoc.

cofibrations on the left: $i^n \cdot_\Delta \mathcal{C}_R \subseteq \mathcal{C}$

$i^n \cdot_\Delta \mathcal{W} \cap \mathcal{C}_e \subseteq \mathcal{W} \cap \mathcal{C}$

Reedy cof. between frames

trivial cofibrations on the left more subtle: $\underbrace{(\Delta^{n-1} \xrightarrow{\sim} \Delta^n)}_{} \cdot_\Delta (X \to Y)$:

$$\begin{tikzcd} X_{n-1} \arrow[r] \arrow[d, "\wr"'] & Y_{n-1} \arrow[d] \arrow[d, "\wr"] \\ X_n \arrow[r] & Y_n \end{tikzcd}$$

these "generate" anodyne extensions in some sense

$\Delta^0 + \Delta^0 = \partial\Delta^1 \rightarrowtail \Delta^1 \xrightarrow{\sim} \Delta^0$ cylinder on Δ^0

$A + A = \underbrace{\partial\Delta^1 \cdot_\Delta A^*}_{L^1 A} \rightarrowtail \underbrace{\Delta^1 \cdot_\Delta A^*}_{A^1} \xrightarrow{\sim} A$ cylinder on A

Balancing. $\mathcal{M}(A^*, X_*) \in s^2Set = [\Delta^{op} \times \Delta^{op}, Set]$

comparison maps:

$$\begin{tikzcd} \mathcal{M}(A^*, X_*) & \mathcal{M}(cst\, A, X_*) \arrow[l] \\ \mathcal{M}(A^*, X) \arrow[u] & \end{tikzcd}$$

ptwise w.e. between Reedy cof. obj's

$\mathcal{M}(A^n, X_*) \xleftarrow{\sim} \mathcal{M}(cst\, A, X_*)$ since $A^n \xrightarrow{\sim} cst\, A$

Take the geometric realizations:

$$\Delta^\bullet \cdot_\Delta \mathcal{M}(A^*, X_*) \xleftarrow{\sim} \Delta^\bullet \cdot_\Delta \mathcal{M}(cst\, A, X_*) = \mathcal{M}(A, X_*)$$

$$\overset{?}{\|}$$

$$diag\, \mathcal{M}(A^*, X_*)$$

then this + the dual statement proves the balancing

this is about bisimplicial sets $K_{\cdot\cdot} = (K_\cdot)_\cdot$.

$$\Delta^\bullet \cdot_\Delta K_\cdot = \int^{n \in \Delta} \underset{sSet}{\Delta^n} \times \underset{sSet}{K_n} = \int^{n \in \Delta} \underset{sSet}{\Delta^n} \times \Big(\int^{k \in \Delta} \underset{sSet}{\Delta^k} \cdot \underset{Set}{K_{nk}} \Big)$$

$$= \int^{n,k} \underset{sSet}{(\Delta^n \times \Delta^k)} \cdot \underset{Set}{K_{nk}} = \int^{(n,k)} \Delta \times \Delta \big(diag\, -, (n,k)\big) \cdot K_{nk}$$

$$\overset{Yoneda}{=} diag\, K$$

Application. $A = (0 \xleftarrow{-} 1 \xrightarrow{+} 2)$.

$$\begin{tikzcd} {[A, \mathcal{M}]} \arrow[r, shift left, "colim"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "cst"] \end{tikzcd}$$

cst pres w.e. (always) and fibrations: $p : X \twoheadrightarrow Y$

$\Rightarrow$ colim takes w.e.'s between Reedy cofibrant

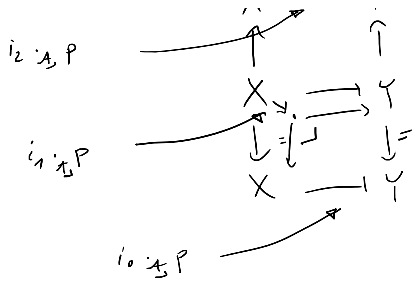
$i_2 \cdot_\Delta p$

$$\begin{tikzcd} X \arrow[r] & Y \\ X_{\cdot\cdot} \arrow[r] \arrow[u] & Y \arrow[u, hook] \end{tikzcd}$$

between Reedy cofibrant diagrams to w.e.'s

(would not work for

$\mathcal{A} = (0 \xleftarrow{=} 1 \xrightarrow{=} 0)$)



Application. $\mathcal{M}$ simplicial

$$[\mathcal{A}^{op}, sSet] \times [\mathcal{A}, \mathcal{M}] \xrightarrow{- \otimes_{\mathcal{A}} -} \mathcal{M}$$ is left Quillen

proj. ptwise   ptwise proj.

$Q* = N(-/\mathcal{A})$ a particular cofibrant replacement of $*$

$\Rightarrow$ $Q* \otimes_{\mathcal{A}} - : [\mathcal{A}, \mathcal{M}_c] \longrightarrow \mathcal{M}_c$ preserves w.e.'s

$$\begin{tikzcd} Q* \otimes_{\mathcal{A}} - \arrow[d] \\ * \otimes_{\mathcal{A}} - \arrow[d, phantom, "\shortparallel"] \\ \operatorname{colim}_{\mathcal{A}} \end{tikzcd}$$

is a w.e. on proj. cof. diagrams

$\Rightarrow$ $N(-/\mathcal{A}) \otimes_{\mathcal{A}} - = \operatorname{hocolim}_{\mathcal{A}}$

This can be extended to all model categories via frames:

$N(-/\mathcal{A}) : \Delta^{op} \times \mathcal{A}^{op} \longrightarrow Set$

$D^* : \Delta \times \mathcal{A} \longrightarrow \mathcal{M}_c$ frame on $D : \mathcal{A} \to \mathcal{M}_c$

$\rightsquigarrow$ $N(-/\mathcal{A}) \cdot_{\Delta \times \mathcal{A}} D^* =: \operatorname{hocolim}_{\mathcal{A}} D$

$\shortparallel$

$N(-/\mathcal{A}) \otimes_{\mathcal{A}} D$ if we denote $K \otimes X = K \cdot_{\Delta} X^*$ the "action" of sSet on $\mathcal{M}$

Bousfield localizations

$\mathcal{M} \rightsquigarrow Ho(\mathcal{M}) \sim \mathcal{M}_{cf}/htpy$

What should a subcategory be ?

We want to limit fibrant / cofibrant objects.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\sim"'] & X \\ B \arrow[ur, dashed] & \end{tikzcd}$$

(homotopy) injectivity — projectivity

$\doteq$ reflective subcat's

Example.

- $Ab \subseteq Gp$

$$\begin{tikzcd} \mathbb{Z} * \mathbb{Z} \arrow[r, "{(a,b)}"] \arrow[d] & A \\ \mathbb{Z} \oplus \mathbb{Z} \arrow[ur, dashed] & \end{tikzcd}$$

$a * b \not\parallel b * a$

$a + b = b + a$

- constant diagrams $D : \Delta \to \mathcal{M}$

$\mathcal{M} = Set$:

$$\begin{tikzcd} \Delta_m \arrow[r] \arrow[d] & D \\ \Delta_n \arrow[ur, dashed, "\exists!"] & \end{tikzcd} \equiv \begin{tikzcd} D_m \\ D_n \arrow[u, "\cong"'] \end{tikzcd}$$

$\mathcal{M}$ general: $\forall X$

$$\begin{tikzcd} \Delta_m \cdot X \arrow[r] \arrow[d] & D \\ \Delta_n \cdot X \arrow[ur, dashed, "\exists!"] & \end{tikzcd} \equiv \begin{tikzcd} \mathcal{M}(X, D_m) \\ \mathcal{M}(X, D_n) \arrow[u, "\cong"'] \end{tikzcd}$$

- diagrams $D : \mathcal{N} \to \mathcal{M}$ st. $D_n \xrightarrow{\cong} D_1 \times \cdots \times D_1$

$\mathcal{M} = Set$:

$$\begin{tikzcd} \mathcal{N}_1 + \cdots + \mathcal{N}_1 \arrow[r] \arrow[d] & D \\ \mathcal{N}_n \arrow[ur, dashed, "\exists!"] & \end{tikzcd} \equiv \begin{tikzcd} D_1 \times \cdots \times D_1 \\ D_n \arrow[u, "\cong"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{N}_1 \arrow[d] \\ \mathcal{N}_n \end{tikzcd} \equiv \begin{tikzcd} 1 \\ n \arrow[u] \end{tikzcd}$$

- diagrams $D : \mathcal{N} \to \mathcal{M}$ st. $D_n \xrightarrow{\cong} \{K, D_{n+1}\}$

$$\begin{tikzcd} \mathcal{N}_{n+1} \otimes K \arrow[r] \arrow[d] & D \\ \mathcal{N}_n \arrow[ur, dashed, "\exists!"] & \end{tikzcd} \begin{tikzcd} \{K, D_{n+1}\} \\ D_n \arrow[u, "\cong"'] \end{tikzcd}$$

$\mathcal{V} = Top_* / sSet_*$

$K = S^1$

$\rightsquigarrow D_n \longrightarrow \Omega D_{n+1}$

$Hom(\mathcal{N}_{n+1} \otimes K, D) = \mathcal{V}(K, Hom(\mathcal{N}_{n+1}, D))$

$= \mathcal{V}(K, D_{n+1})$

- sheaves

$$\begin{tikzcd} \operatorname{colim} \mathcal{X}_{U_i} \arrow[r] \arrow[d] & D \\ \mathcal{X}_U \arrow[ur, dashed, "\exists!"] & \end{tikzcd} \begin{tikzcd} \lim D_{U_i} \\ D_U \arrow[u, "\cong"'] \end{tikzcd}$$

$\mathcal{X} = Op(X)$

U_i open covering of U

closed under intersections

Example. where htpy version is necessary

$\mathcal{M} = Top$:

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X \\ D^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

$\exists$ $\underbrace{\text{unique up to higher htpies}}_{?}$

$\pi_n X = 0$

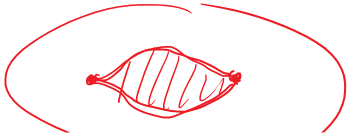
X is a homotopy (n-1)-type

$$\begin{tikzcd} map(S^n, X) \\ map(D^{n+1}, X) \arrow[u, "\sim"'] \end{tikzcd}$$

$\pi_n X = 0$

$\pi_{n+1} X = 0$

$\vdots$

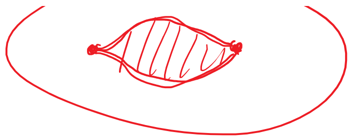


more generally

$$\begin{tikzcd} A \arrow[r] \arrow[d, dash] & X \\ {} \arrow[ur] & \end{tikzcd}$$

$map(A, X)$

$\uparrow$



more generally
$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ B \arrow[ur, dashed, "\exists!_h"'] & \end{tikzcd} \equiv \begin{tikzcd} map(A,X) \\ map(B,X) \arrow[u, two heads, "\sim"'] \end{tikzcd}$$

better:
$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \\ B \arrow[ur, dashed, "\exists!_h"'] & \end{tikzcd} \equiv \begin{tikzcd} map(A,X) & L \arrow[l] \arrow[dl, dashed] \\ map(B,X) \arrow[u, two heads, "\sim"'] & K \arrow[l] \arrow[u, tail] \end{tikzcd} \equiv \begin{tikzcd} K \times B +_{K \times A} L \times A \arrow[r] \arrow[d] & X \\ L \times B \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

cofibration (between cofibrant)

in Top $S^{n-1} \rightarrowtail D^n$

Will assume $\mathcal{M}$ left proper, cellular = cofibrantly generated with cofibrations effective mono's

Definition. Let $f: A \to B$ be a cofibration between cofibrant objects. We say that W is f-local if it is fibrant and

$$f^*: map(B,W) \xrightarrow{\sim} map(A,W)$$

is a weak equivalence of simplicial sets.

$\Big\} \left(\mathcal{J} \cup \{L^n B +_{L^n A} A^n \longrightarrow B^n\}\right)^{\square}$

$n=0$: $A \longrightarrow B$

Definition. A map $g: X \to Y$ is an f-local equivalence if its cofibrant replacement $\tilde{g}: \tilde{X} \rightarrowtail \tilde{Y}$

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, equal] & \tilde{X} \arrow[r, "\sim"] \arrow[d, tail, "\tilde{g}"] & X \arrow[d, "g"] \\ 0 \arrow[r, tail] & \tilde{Y} \arrow[r, "\sim"] & Y \end{tikzcd}$$

$\Rightarrow$ f is an f-local equiv.

gives, for each f-local W, a w.e. $\tilde{g}^*: map(\tilde{Y}, W) \xrightarrow{\sim} map(\tilde{X}, W)$

(any two related by a zig-zag of w.e.'s of such $\Rightarrow$ independent of choice)

Definition. An f-localization of X is an f-local equivalence
$j: X \longrightarrow \hat{X}$ with $\hat{X}$ f-local.

Aim. Construct an "f-local" model structure in which:

- f-local = fibrant ; fibrations are complicated BUT cofibrations of $\mathcal{M}$
- f-local equivalence = weak equivalence
- f-localization = fibrant replacement = "reflection"

$\to$ better: $\underset{\text{preserves cof. \& w.e.}}{Id}: \mathcal{M} \rightleftarrows \mathcal{M}^{f\text{-local}} : Id$

is interpreted as $\mathcal{M}_{cf} \underset{f\text{-localization}}{\rightleftarrows} \mathcal{M}^{f\text{-local}}_{cf}$