(a) Show that A is independent of B, and that B is independent of C.

(b) Is A independent of C?

(c) Do these results hold if boys and girls are not equally likely?

(d) Do these results hold if Jane has four children?

8. Galton's paradox. You flip three fair coins. At least two are alike, and it is an evens chance that the third is a head or a tail. Therefore $\mathbb{P}(\text{all alike}) = \frac{1}{2}$. Do you agree?

9. Two fair dice are rolled. Show that the event that their sum is 7 is independent of the score shown by the first die.

1.7 Exercises. Worked examples

1. There are two roads from A to B and two roads from B to C. Each of the four roads is blocked by snow with probability p, independently of the others. Find the probability that there is an open road from A to B given that there is no open route from A to C.

If, in addition, there is a direct road from A to C, this road being blocked with probability p independently of the others, find the required conditional probability.

2. Calculate the probability that a hand of 13 cards dealt from a normal shuffled pack of 52 contains exactly two kings and one ace. What is the probability that it contains exactly one ace given that it contains exactly two kings?

3. A symmetric random walk takes place on the integers $0, 1, 2, \ldots, N$ with absorbing barriers at 0 and N, starting at k. Show that the probability that the walk is never absorbed is zero.

4. The so-called 'sure thing principle' asserts that if you prefer x to y given C, and also prefer x to y given C^c, then you surely prefer x to y. Agreed?

5. A pack contains m cards, labelled $1, 2, \ldots, m$. The cards are dealt out in a random order, one by one. Given that the label of the kth card dealt is the largest of the first k cards dealt, what is the probability that it is also the largest in the pack?

1.8 Problems

1. A traditional fair die is thrown twice. What is the probability that:

(a) a six turns up exactly once?

(b) both numbers are odd?

(c) the sum of the scores is 4?

(d) the sum of the scores is divisible by 3?

2. A fair coin is thrown repeatedly. What is the probability that on the nth throw:

(a) a head appears for the first time?

(b) the numbers of heads and tails to date are equal?

(c) exactly two heads have appeared altogether to date?

(d) at least two heads have appeared to date?

3. Let $\mathcal{F}$ and $\mathcal{G}$ be σ-fields of subsets of Ω.

(a) Use elementary set operations to show that $\mathcal{F}$ is closed under countable intersections; that is, if $A_1, A_2, \ldots$ are in $\mathcal{F}$, then so is $\bigcap_i A_i$.

(b) Let $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ be the collection of subsets of Ω lying in both $\mathcal{F}$ and $\mathcal{G}$. Show that $\mathcal{H}$ is a σ-field.

(c) Show that $\mathcal{F} \cup \mathcal{G}$, the collection of subsets of Ω lying in either $\mathcal{F}$ or $\mathcal{G}$, is not necessarily a σ-field.

4. Describe the underlying probability spaces for the following experiments:

(a) a biased coin is tossed three times;

(b) two balls are drawn without replacement from an urn which originally contained two ultramarine and two vermilion balls;

(c) a biased coin is tossed repeatedly until a head turns up.

5. Show that the probability that *exactly* one of the events A and B occurs is

$$\mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B).$$

6. Prove that $\mathbb{P}(A \cup B \cup C) = 1 - \mathbb{P}(A^c \mid B^c \cap C^c)\mathbb{P}(B^c \mid C^c)\mathbb{P}(C^c)$.

7. (a) If A is independent of itself, show that $\mathbb{P}(A)$ is 0 or 1.

(b) If $\mathbb{P}(A)$ is 0 or 1, show that A is independent of all events B.

8. Let $\mathcal{F}$ be a σ-field of subsets of Ω, and suppose $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfies: (i) $\mathbb{P}(\Omega) = 1$, and (ii) $\mathbb{P}$ is additive, in that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ whenever $A \cap B = \varnothing$. Show that $\mathbb{P}(\varnothing) = 0$.

9. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $B \in \mathcal{F}$ satisfies $\mathbb{P}(B) > 0$. Let $\mathbb{Q} : \mathcal{F} \to [0, 1]$ be defined by $\mathbb{Q}(A) = \mathbb{P}(A \mid B)$. Show that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space. If $C \in \mathcal{F}$ and $\mathbb{Q}(C) > 0$, show that $\mathbb{Q}(A \mid C) = \mathbb{P}(A \mid B \cap C)$; discuss.

10. Let $B_1, B_2, \ldots$ be a partition of the sample space Ω, each B_i having positive probability, and show that

$$\mathbb{P}(A) = \sum_{j=1}^{\infty} \mathbb{P}(A \mid B_j)\mathbb{P}(B_j).$$

11. Prove **Boole's inequalities**:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \mathbb{P}(A_i), \qquad \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \ge 1 - \sum_{i=1}^{n} \mathbb{P}(A_i^c).$$

12. Prove that

$$\mathbb{P}\left(\bigcap_{1}^{n} A_i\right) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cup A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cup A_j \cup A_k) - \cdots - (-1)^n \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n).$$

13. Let $A_1, A_2, \ldots, A_n$ be events, and let N_k be the event that exactly k of the A_i occur. Prove the result sometimes referred to as **Waring's theorem**:

$$\mathbb{P}(N_k) = \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{k} S_{k+i}, \quad \text{where } S_j = \sum_{i_1 < i_2 < \cdots < i_j} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}).$$

Use this result to find an expression for the probability that a purchase of six packets of Corn Flakes yields exactly three distinct busts (see Exercise (1.3.4)).

14. Prove **Bayes's formula**: if $A_1, A_2, \ldots, A_n$ is a partition of Ω, each A_i having positive probability, then

$$\mathbb{P}(A_j \mid B) = \frac{\mathbb{P}(B \mid A_j)\mathbb{P}(A_j)}{\sum_1^n \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)}.$$