4.1. $f : \mathbb{R}^3 \to \mathbb{R}^2$

$$f\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \qquad f\begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$$

(a) určete $f\begin{pmatrix}8\\13\\18\end{pmatrix}$

Pozn.: f je zadáno pouze na dvou LN vektorech, tj. není určeno jednoznačně jako zobr. $f : \mathbb{R}^3 \to \mathbb{R}$

$$a\begin{pmatrix}1\\2\\3\end{pmatrix} + b\begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}8\\13\\18\end{pmatrix}$$

$$\boxed{\begin{aligned} a + 2b &= 8 \quad /\cdot(-2) \\ 2a + 3b &= 13 \end{aligned}}$$

$$3a + 4b = 18$$

$$-b = 13 - 16 = -3$$

$$\boxed{b = 3}$$

$$a + 2\cdot 3 = 8$$

$$\boxed{a = 2}$$

$$f\begin{pmatrix}8\\13\\18\end{pmatrix} = f\left(2\begin{pmatrix}1\\2\\3\end{pmatrix} + 3\begin{pmatrix}2\\3\\4\end{pmatrix}\right) =$$

$$= 2\cdot f\begin{pmatrix}1\\2\\3\end{pmatrix} + 3 f\begin{pmatrix}2\\3\\4\end{pmatrix} =$$

$$= 2\cdot\begin{pmatrix}2\\1\end{pmatrix} + 3\cdot\begin{pmatrix}2\\4\end{pmatrix} =$$

$$= \underline{\underline{\begin{pmatrix}10\\14\end{pmatrix}}}$$

<u>Jiná varianta:</u>

$$f\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \qquad f\begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$$

+ na bázi $\mathbb{R}^3$.

Přidám: $f\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$

<u>Určíme jádro</u> $f: \mathbb{R}^3 \to \mathbb{R}^2$:

① $f\left(a\begin{pmatrix}1\\2\\3\end{pmatrix}+b\begin{pmatrix}2\\3\\4\end{pmatrix}+c\begin{pmatrix}1\\0\\1\end{pmatrix}\right)=0$

$a\,f\begin{pmatrix}1\\2\\3\end{pmatrix}+b\cdot f\begin{pmatrix}2\\3\\4\end{pmatrix}+c\,f\begin{pmatrix}1\\0\\1\end{pmatrix}=0$

$a\begin{pmatrix}2\\1\end{pmatrix}+b\begin{pmatrix}2\\4\end{pmatrix}+c\begin{pmatrix}1\\3\end{pmatrix}=0$

$2a+2b+c=0$

$a+4b+3c=0 \;/\cdot(-2)$ $\qquad p\in\mathbb{R}$

$-6b-5c=0$ $\qquad \boxed{c=6p}$

$6b+5\cdot 6p=0$ $\qquad \boxed{b=-5p}$

$a+4(-5p)+3\cdot 6p=0$ $\qquad \boxed{a=2p}$

$a-2p=0$

$p=1$: $2\begin{pmatrix}1\\2\\3\end{pmatrix}-5\begin{pmatrix}2\\3\\4\end{pmatrix}+6\begin{pmatrix}1\\0\\1\end{pmatrix}=$

$=\begin{pmatrix}2-10+6\\4-15\\6-20+6\end{pmatrix}=\begin{pmatrix}-2\\-11\\-8\end{pmatrix}$

Závěr: $\operatorname{Ker} f=\left\langle\begin{pmatrix}2\\11\\8\end{pmatrix}\right\rangle$

Matice zobr. f :

$$\alpha = \left(\underset{v_1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \right) \quad \text{báze } \mathbb{R}^3$$

$$\epsilon = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \quad \text{std. báze } \mathbb{R}^2$$

$$(f)_{\epsilon,\alpha} = \left((f(v_1))_\epsilon, (f(v_2))_\epsilon, (f(v_3))_\epsilon \right)$$

$$= \begin{pmatrix} 2 & 2 & 1 \\ 1 & 4 & 3 \end{pmatrix}$$

$$(f)_{\epsilon,\epsilon} = \underline{(f)_{\epsilon,\alpha}} \cdot \underbrace{(id)_{\alpha,\epsilon}}$$

$$(id)_{\epsilon,\alpha} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 3 & 4 & 1 \end{pmatrix}, \quad (id)_{\alpha,\epsilon} = \left((id)_{\epsilon,\alpha} \right)^{-1}$$

4.12 $C = \mathcal{J} = \left(\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$\mathcal{B} = \mathcal{B} = \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$L : \mathbb{R}^3 \to \mathbb{R}^3,$

$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x+z \\ x+z \end{pmatrix} \leftarrow$ predpis

Určete bázi $\ker L$, $\operatorname{im} L$, a určete $(L)_{\mathcal{B},\mathcal{J}}$

- Jádro L: $L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x+z \\ x+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left.\begin{aligned} x+y &= 0 \\ x+z &= 0 \end{aligned}\right\} \quad \begin{aligned} y &= -x \\ z &= -x \end{aligned} \quad \Rightarrow$

$\Rightarrow (x, y, z) = p\,(1, -1, -1)$

$\ker L = \left\langle \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\rangle, \quad \dim(\ker L) = 1$

- Obraz: $\dim(\operatorname{Im} L) = 2$

$$\left.\begin{array}{l} L\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \\ L\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \\ L\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} \end{array}\right\} \text{lin. zależne wektory}$$

$$\operatorname{im} L = \Big\langle \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}}_{\text{baza obrazu}} \Big\rangle$$

$$L\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y\\x+z\\x+z\end{pmatrix}$$

$$(L)_{E,E} = \left(\left(L\begin{pmatrix}1\\0\\0\end{pmatrix}\right)_E, \left(L\begin{pmatrix}0\\1\\0\end{pmatrix}\right)_E, \left(L\begin{pmatrix}0\\0\\1\end{pmatrix}\right)_E \right)$$

$$= \begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & 0 & 1\end{pmatrix}$$

$$(L)_{B,\mathcal{F}} = (id)_{B,E} \cdot (L)_{E,E} \cdot (id)_{E,\mathcal{F}}$$

$$(id)_{E,\mathcal{F}} = \begin{pmatrix}2 & 1 & 0\\ 3 & 2 & 0\\ 1 & 3 & 1\end{pmatrix} \qquad (id)_{B,E}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$(id)_{E,B} = \begin{pmatrix}3 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix} \qquad \left((id)_{E,B}\right)^{-1}$$

Př 5: Symetrie podle roviny
$x - 2z + 3 = 0$

- $\alpha = (v_1, v_2, n)$ $\quad n = (1, -2, 3)$

$v_1 \perp n$ $\quad v_1 = (2, 1, 0)$

$v_2 \perp n$ $\quad v_2 = (3, 0, -1)$

- $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ symetrie podle zadané roviny

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(\varphi)_{\epsilon,\epsilon} = (id)_{\epsilon,\alpha} \cdot (\varphi)_{\alpha,\alpha} \cdot (id)_{\alpha,\epsilon}$$