6.2
$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & 1 & 1 \end{array}\right) \sim$$

$-a$ $\quad$ -1

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -a+1 & -a+1 & -a+1 \\ 0 & a-1 & 0 & a-1 \end{array}\right) \begin{array}{c} \frac{1}{a-1} \\ \frac{1}{a-1} \end{array}$$

$a \neq 1$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\boxed{\begin{array}{l} x_3 = 0 \\ x_2 = 1 \\ x_1 = 0 \end{array}}$$

$x_1 + x_2 + x_3 = 1$

$x_1 + 1 + 0 = 1$

$a = 1$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right)$$

$x_2 = p$ $\quad$ $x_3 = q$

$x_1 = 1 - p - q$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + p \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + q \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

<u>Závěr:</u> • $a = 1 \Rightarrow$ nekon. mnoho řešení

• $a \neq 1 \Rightarrow x_1 = x_3 = 0,\ x_2 = 1$ právě jedno řešení

61) $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & -3 \\ 2 & 1 & 2 \end{pmatrix}$, $A^{-1} = ?$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & -3 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim$$

-1, -2

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -5 & -1 & 1 & 0 \\ 0 & -1 & -2 & -2 & 0 & 1 \end{array}\right)$$

(-2)

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & -2 \end{array}\right) \begin{array}{l} \\ 2 \\ 2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 7 & 2 & -4 \\ 0 & 1 & 0 & 8 & 2 & -5 \\ 0 & 0 & 1 & -3 & -1 & 2 \end{array}\right) (-1)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 8 & 2 & -5 \\ 0 & 0 & 1 & -3 & -1 & 2 \end{array}\right)$$

$$A^{-1}$$

$$A \cdot A^{-1} = \ldots$$

6.3 $v_1 = (1,1,1,2)$

$v_2 = (-1,-1,1,2)$ $\in \mathbb{R}^4$

$v_3 = (1,1,3,6)$

(i) $v_4 = (3,3,1,2)$

$$\begin{array}{c} \\ -1 \\ -1 \\ -2 \end{array}\begin{pmatrix} 1 & -1 & 1 & 3 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & 3 & 1 \\ 2 & 2 & 6 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 4 & -4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

Závěr: $\alpha = (v_1, v_2)$ je <u>báze</u> podprostoru $\langle v_1, v_2, v_3, v_4 \rangle$.

Dále doplníme α na bázi $\mathbb{R}^4$:

$$\left(\begin{array}{cc|cccc} 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \sim \quad -2$$

$$\underbrace{v_1 \quad v_2}_{\text{base } \alpha} \qquad \underbrace{\qquad\qquad}_{\text{est. base } \mathbb{R}^4}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 4 & -2 & 0 & 0 & 1 \end{pmatrix} \sim \quad -2$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{pmatrix}$$

$$\mathcal{B} = \left(v_1, v_2, \underbrace{(1,0,0,0)}_{\substack{E_1 \\ E_1}}, \underbrace{(0,0,1,0)}_{\substack{E_3 \\ E_3}}\right)$$

(ii) Určete souřadnice vektoru $u = (5,4,2,4) \in \mathbb{R}^4$ v bázi B

$a_1, a_2, a_3, a_4 \in \mathbb{R}$ t. ž.

$$a_1 v_1 + a_2 v_2 + a_3 e_1 + a_4 e_3 = u$$

$$\boxed{(u)_B = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4}$$

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 5 \\ 1 & -1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 0 & 4 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 5 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & 1 & -3 \\ 0 & 4 & -2 & 0 & -6 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 5 \\ 0 & 2 & -1 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right)$$

$2a_2 - a_3 = -3$ | $\boxed{a_4 = 0}$ | $\boxed{a_3 = 1}$

$2a_2 = -2$

$\boxed{a_2 = -1}$

$a_1 - a_2 + a_3 = 5$

$a_1 - (-1) + 1 = 5$

$\boxed{a_1 = 3}$

Zinker: $(u)_B = (3, -1, 3, 0)$

6.4 (i) $M = \left\{ \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

$$\subseteq Mat_{2,2}(\mathbb{R})$$

$M \subseteq V$ je podprostor

↳ podmn. ↳ vektorový prostor

jestliže $v_1, v_2 \in M$ a t.

platí:

- $v_1 + v_2 \in M$
- $p \cdot v_1 \in M \quad \forall p \in \mathbb{R}$

$$\begin{pmatrix} a_1 & 1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & 1 \\ 0 & b_2 \end{pmatrix} \in M$$

Pak $\begin{pmatrix} a_1 & 1 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & 2 \\ 0 & b_1 + b_2 \end{pmatrix} \notin M$

$\Longrightarrow$ M není podprostor

$$M' = \left\{ \begin{pmatrix} a & 0 \\ a+b & b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$$

$$\subseteq Mat_{2,2}(\mathbb{R})$$

$$\begin{pmatrix} a_1 & 0 \\ a_1+b_1 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ a_2+b_2 & b_2 \end{pmatrix} \in M'$$

- $$\begin{pmatrix} a_1 & 0 \\ a_1+b_1 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ a_2+b_2 & b_2 \end{pmatrix} =$$

$$= \begin{pmatrix} a_1+a_2 & 0 \\ (a_1+a_2)+(b_1+b_2) & b_1+b_2 \end{pmatrix} \in M'$$

- $$p \begin{pmatrix} a_1 & 0 \\ a_1+b_1 & b_1 \end{pmatrix} = \begin{pmatrix} pa_1 & 0 \\ pa_1+pb_1 & pb_1 \end{pmatrix}$$

$$\in M'$$

Záver: M' je podpriestor v $Mat_{2,2}(\mathbb{R})$

báze M' : $\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right)$

$\dim M' = 2$

(ii) $Q = \{ f(x) \mid f(1) = 0 \wedge f(2) = 0 \}$
$\subseteq \mathbb{R}_4[x]$

$f, f' \in Q \Rightarrow f(1) = f(2) = 0$
$f'(1) = f'(2) = 0$

- $(f+f')(1) = f(1) + f'(1) = 0$
  $\hookrightarrow$ definice součtu polynomů

$(f+f')(2) = \text{—} \cdot \text{—} \cdot = 0$

$\Rightarrow f + f' \in Q$

- $p \in \mathbb{R}$: $(pf)(1) = p \cdot f(1) = 0$
  $(pf)(2) = \qquad = 0$

$\Longrightarrow pf \in Q$

Závěr: Q je podprostor

Podobně S je podprostor

Báze Q:

$f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$
$\cap$
$\mathbb{R}_4[x]$

$a_i \in \mathbb{R}$

$f(1) = 0 \Rightarrow a_4 + a_3 + a_2 + a_1 + a_0 = 0$

$f(2) = 0 \Rightarrow 16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0 = 0$

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & a_4 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 15 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & -2 & -6 & -14 \\ 0 & 1 & 3 & 7 & 15 \end{pmatrix}$$

$a_2 = p, \quad a_3 = q, \quad a_4 = r$

$a_1 = -3a_2 - 7a_3 - 15a_4$

$$a_0 = 2\overset{p}{a_2} + 6\overset{q}{a_3} + 14\overset{r}{a_4}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = p\begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + q\begin{pmatrix} 6 \\ -7 \\ 0 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 14 \\ -15 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\dim Q = 3}$$

- $p = 1, \; q = r = 0 \rightsquigarrow \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\boxed{f_1 = x^2 - 3x + 2}$$

- $q = 1, \; p = r = 0 \rightsquigarrow \begin{pmatrix} 6 \\ -7 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$\boxed{f_2 = x^3 - 7x + 6}$$

- $r = 1, \; p = q = 0 \rightsquigarrow \begin{pmatrix} 14 \\ -15 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$\boxed{f_3 = x^4 - 15x + 14}$$

Závěr: $\alpha = (f_1, f_2, f_3)$

je báze Q

Jiný postup:

$f(x) \in Q \iff$

$\iff f(x) = (x-1)(x-2)(ax^2+bx+c)$

$a, b, c \in \mathbb{R}$

Jiná báze:

$\alpha' = \big(x^2(x-1)(x-2),\ x(x-1)(x-2),$

$(x-1)(x-2) \big)$