6.4 (ii). $Q, S \subseteq \mathbb{R}_4[x]$

Báze Q: $\alpha = (f_1, f_2, f_3)$, kde

$$f_1 = x^2 - 3x + 2$$
$$f_2 = x^3 - 7x + 6$$
$$f_3 = x^4 - 15x + 14$$

$$S = \{g(x) \mid g(x) = g(-x)\}$$
$$B = (g_1, g_2, g_3)$$
$$g_1 = 1, \quad g_2 = x^2$$
$$g_3 = x^4$$

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in S$$
$$\|$$
$$a_4x^4 - a_3x^3 + a_2x^2 - a_1x + a_0 \Rightarrow 2a_3x^3 + 2a_1x = 0$$
$$\Rightarrow a_3 = a_1 = 0$$

6.5 Spočteme $Q \cap S$

$r(x) \in Q \cap S \iff$

$$r(x) = a_1f_1 + a_2f_2 + a_3f_3 = b_1g_1 + b_2g_2 + b_3g_3$$

$$a_1\begin{pmatrix}0\\0\\1\\-3\\2\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\\-7\\6\end{pmatrix} + a_3\begin{pmatrix}1\\0\\0\\-15\\14\end{pmatrix} - b_1\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} - b_2\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} - b_3\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = 0$$

$$\begin{array}{c}\\ \\ \\ \\ -2\end{array}\begin{pmatrix}0 & 0 & 1 & 0 & 0 & -1\\0 & 1 & 0 & 0 & 0 & 0\\1 & 0 & 0 & 0 & -1 & 0\\-3 & -7 & -15 & 0 & 0 & 0\\2 & 6 & 14 & -1 & 0 & 0\end{pmatrix}\begin{array}{c}\\ \\ 3\\ \\ \\ \end{array}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -7 & -15 & 0 & -3 & 0 \\ 0 & 6 & 14 & -1 & 2 & 0 \end{pmatrix} \begin{matrix} \\ \\ 7 \\ \\ -6 \end{matrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -15 & 0 & -3 & 0 \\ 0 & 0 & 14 & -1 & 2 & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ 15 \\ -14 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -3 & -15 \\ 0 & 0 & 0 & -1 & 2 & 14 \end{pmatrix} \sim \begin{array}{cccccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ \hline \end{array}$$

$$\begin{array}{c} \\ \end{array}\begin{array}{cccccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 & -14 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix}$$

$\dim(Q \cap S) = 1$

$b_3 = p \in \mathbb{R}$

$b_2 = -5p$

$b_1 = 4p$

$a_3 = p$

$a_2 = 0$

$a_1 = -5p$

$b_1 - 2(-5p) - 14p = 0$

$b_1 - 4p = 0$

$b_2 + 5p = 0$

- $r(x) = -5p f_1 + p f_3 =$

$= p(-5f_1 + f_3) = p(x^4 - 5x^2 + 4)$

Závěr: $Q \cap S$ je generovaný polynomem $x^4 - 5x^2 + 4$.

↳ báze $Q \cap S$

$$\underbrace{\dim Q}_{3} + \underbrace{\dim S}_{3} = \underbrace{\dim(Q+S)}_{\substack{5 \\ \text{\rotatebox{90}{$\leq$}} \\ \dim \mathbb{R}_4[x]}} + \underbrace{\dim(Q \cap S)}_{1}$$

$$\boxed{Q + S = \mathbb{R}_4[x]} \Leftarrow \dim \mathbb{R}_4[x]$$

7.1: $v_1 = (1,1,1,1)$ $\quad V = \langle v_1, v_2, v_3 \rangle \leq \mathbb{R}^4$

$v_2 = (1,0,0,3)$ $\quad$ Chceme bázi

$v_3 = (1,2,1,0)$ $\quad \alpha = (w_1, w_2, w_3)$ t.ž.

$0 = \langle w_2, w_3 \rangle = \langle w_1, w_2 \rangle = \langle w_1, w_3 \rangle$

tj. ortogonální bázi.

$$\boxed{w_1 := v_1 = (1,1,1,1)}$$

$w_2 := v_2 + a v_1$, $a \in \mathbb{R}$ splňuje

$$\langle w_2, w_1 \rangle = \langle v_2 + a v_1, v_1 \rangle = 0$$

$$\langle v_2, v_1 \rangle + \langle a v_1, v_1 \rangle = 0$$

$$\langle v_2, v_1 \rangle + a \langle v_1, v_1 \rangle = 0$$

$4 + 4a = 0$

$\underbrace{\qquad}_{4} \qquad \underbrace{\qquad}_{4}$

$a = -1$

$$\boxed{w_2 = v_2 - v_1 = (0, -1, -1, 2)}$$

$a, b \in \mathbb{R}$

$w_3 = v_3 + a v_1 + b v_2$

$\langle w_3, w_1 \rangle = \langle v_3 + a v_1 + b v_2, w_1 \rangle = 0$

$\langle w_3, w_2 \rangle = \langle v_3 + a v_1 + b v_2, w_2 \rangle = 0$

$\langle w_3, w_1 \rangle = \langle v_3, v_1 \rangle + a \langle v_1, v_1 \rangle + b \langle w_2, v_1 \rangle^{=0}$

$4 + 4a + 4b = 0$

$a + b + 1 = 0$

$\langle w_3, w_2 \rangle = \langle v_3, w_2 \rangle + a \langle v_1, w_2 \rangle + b \langle w_2, w_2 \rangle = 0$

$-3 + 0 + 6b = 0$

$2b - 1 = 0 \qquad b = \frac{1}{2}$

$a = -\frac{3}{2}$

$w_3 = v_3 - \frac{3}{2} v_2 + \frac{1}{2} v_1$

$= (1, 2, 1, 0) - \frac{3}{2}(1, 1, 1, 1) + \frac{1}{2}(1, 0, 0, 3)$

$= (0, \frac{1}{2}, -\frac{1}{2}, 0)$

$$\boxed{w_3 = (0, \frac{1}{2}, -\frac{1}{2}, 0)}$$

$B = \langle u_1, u_2, u_3 \rangle$ ortonormální báze

norma $\|u_i\| = \sqrt{\langle u_i, u_i \rangle} = 1$

$u_1 := c_1 w_1 \quad , \quad c_1 \in \mathbb{R}, \quad c_1 > 0$

$\|u_1\| = \|c_1 w_1\| = |c_1| \cdot \underbrace{\|w_1\|} = 1 \Rightarrow c_1 = \frac{1}{2}$

$\sqrt{1+1+1+1} = \sqrt{4} = 2$

$$\boxed{u_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}$$

$u_2 = c_2 w_2 \quad , \quad c_2 > 0$

$\|u_2\| = c_2 \underbrace{\|w_2\|} = c_2 \cdot \sqrt{6} = 1$

$c_2 = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$

$$\boxed{u_2 = \left(0, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)}$$

$u_3 = c_3 w_3, \quad c_3 > 0$

$\|u_3\| = c_3 \underbrace{\|w_3\|} = 1 \quad \Rightarrow c_3 = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$

$$\boxed{u_3 = \left(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)}$$

7.2. $\varphi: \mathbb{R}_3[x] \to \mathbb{R}^2$

$$\varphi(f) = (f(1), f(2))$$

<u>Obecně</u>: $\varphi: V_1 \to V_2$

báze α_1 báze α_2

$$\boxed{(\varphi)_{\alpha_2,\alpha_1} = \left((\varphi(v_1))_{\alpha_2} \ (\varphi(v_2))_{\alpha_2} \ \dots \ (\varphi(v_k))_{\alpha_2} \right)}$$

matice lin. zob. φ v bázi α_1 do báze α_2

typu l/k

$\alpha_1 = (v_1, \dots, v_k)$

$\alpha_2 = (w_1, \dots, w_l)$

dim 4   dim 2

$$\varphi: \mathbb{R}_3[x] \to \mathbb{R}^2$$

$$\varphi(f) \mapsto (f(1), f(2))$$

$\alpha_1 = (x^3, x^2, x, 1)$ $\leftarrow E, E$

$\alpha_2 = ((1,0), (0,1))$

$$(\varphi(x^3))_{\alpha_2} = (1,8)_{\alpha_2} = (1,8)$$

$$(\varphi(x^2))_{\alpha_2} = (1,4)_{\alpha_2} = (1,4)$$

$(\varphi(x))_{\alpha_2} = (1,2)_{\alpha_2} = (1,2)$

$(\varphi(1))_{\alpha_2} = (-1,1)_{\alpha_2} = (1,1)$

$$(\varphi)_{\alpha_2,\alpha_1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{pmatrix}$$

(a) $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ lin. zob

$\varphi(1,0,1) = (0,1,0)$

$\to \varphi(0,1,0) = (0,0,1)$

$\to \varphi(0,0,1) = (1,0,1)$

$\varphi(a_1 v_1 + a_2 v_2) = a_1 \varphi(v_1) + a_2 \varphi(v_2)$

linearita

Stand. báze $\epsilon = ((1,0,0),(0,1,0),(0,0,1))$

$(1,0,0) = (1,0,1) - (0,0,1)$

$$\begin{aligned}\varphi(1,0,0) &= \varphi((1,0,1) - (0,0,1)) \\ &= \varphi(1,0,1) - \varphi(0,0,1) \\ &= (0,1,0) - (1,0,1) = (-1,1,-1)\end{aligned}$$

$\varphi(0,1,0) = (0,0,1)$

$\varphi(0,0,1) = (1,0,1)$

$$(\varphi)_{\epsilon,\epsilon} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Jiný postup: $\epsilon = \dots$

$\alpha = ((1,0,1),(0,1,0),(0,0,1))$

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

navzájem inverzní matice

$$(\varphi)_{\epsilon,\epsilon} = \underbrace{(id)_{\epsilon,\alpha}}_{\text{matice přechodu z } \alpha \text{ do } \epsilon} \cdot (\varphi)_{\alpha,\alpha} \cdot \underbrace{(id)_{\alpha,\epsilon}}_{\text{matice přechodu z } \epsilon \text{ do } \alpha}$$

$$(id)_{\epsilon,\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$(id)_{\alpha,\epsilon} = \left((id)_{\epsilon,\alpha}\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$(\varphi)_{\epsilon,\epsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

(b) $$(\varphi)_{\epsilon,\epsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

побот $\varphi(a, b, c) = (a, b, 0)$