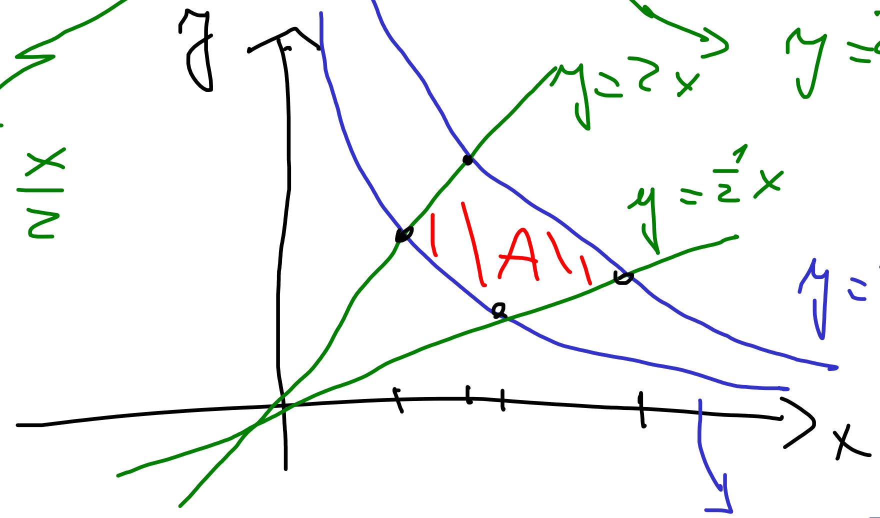
5.5 $I = \iint\limits_A x^2 y^2 \, dx\, dy$, kde

mn. A je ohraničena křivkami

$xy = \frac{1}{2}$, $xy = 2$, $2y = x$, $2x = y$

$y = \frac{1}{2x}$ $y = \frac{2}{x}$ $y = \frac{x}{2}$ $y = 2x$



$y = \frac{1}{2x}$

$u = xy$

$\frac{1}{2} \le u \le 2$

u, v nové souřadnice

$y = vx$

$\frac{1}{2} \le v \le 2$

$x^2 y^2 = u^2$

$xy = u$

$(x, y) = G(u, v)$, kde

$$G(u,v) = \Big(\underbrace{\sqrt{\frac{u}{v}}}_{x}, \underbrace{\sqrt{uv}}_{y}\Big)$$

$u = xy$, $y = vx$

$u = v x^2$, $x = \sqrt{\frac{u}{v}}$

$u = x \cdot y = \frac{y}{v} \cdot y$

$\sqrt{uv} = y$

$$G'(u,v) = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix}$$

$$\det G' = \frac{1}{4}\frac{1}{v} + \frac{1}{4}\frac{1}{v} = \frac{1}{2v}$$

$$I = \int_{u=\frac{1}{2}}^{u=2} \int_{v=\frac{1}{2}}^{2} u^2 \cdot \frac{1}{2v} \, du\, dv =$$

$$= \left(\int_{1/2}^{2} u^2 du\right)\left(\int_{1/2}^{2} \frac{1}{2v} dv\right) =$$

$$= \left[\frac{1}{3}u^3\right]_{1/2}^{2} \left[\frac{1}{2}\ln v\right]_{1/2}^{2} =$$

$$= \frac{1}{3}\left(8 - \frac{1}{8}\right) \cdot \frac{1}{2}\left(\ln 2 - \ln 1/2\right) =$$

$$= \frac{1}{6}\,\frac{64-1}{8} \cdot (\ln 2 + \ln 2) = \underline{\underline{\frac{63}{24}\ln 2}}$$

5.6 Určete obsah rovinného obrazce omezeného křivkami

$x=0, \quad y=\frac{1}{x}, \quad y=8, \quad y=4x$

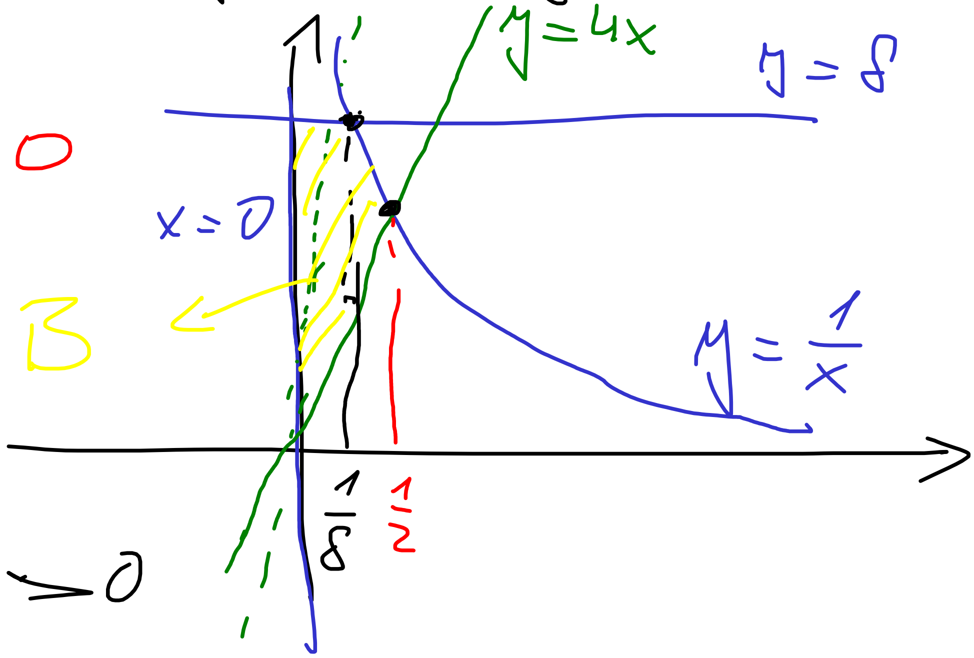
$y=4x=\frac{1}{x}, \quad x>0$

$x^2=\frac{1}{4}$

$x=\frac{1}{2}$

$y=8=\frac{1}{x}, \quad x>0$

$x=\frac{1}{8}$



$$S=\iint\limits_{B} dx\,dy=$$

$$=\int_0^{1/8}\int_{4x}^{8} dy\,dx+\int_{1/8}^{1/2}\int_{4x}^{1/x} dy\,dx$$

$$=\int_0^{1/8}\left[y\right]_{4x}^{8}dx+\int_{1/8}^{1/2}\left[y\right]_{4x}^{1/x}dx$$

$$=\int_0^{1/8}(8-4x)\,dx+\int_{1/8}^{1/2}(1/x-4x)\,dx$$

$$= 4\left[2x - \frac{1}{2}x^2\right]_0^{1/8} + \left[\ln x - 2x^2\right]_{1/8}^{1/2}$$

$$= 4\left[\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{64}\right] + \left[\ln(1/2) - \frac{2}{4} - \left(\ln\frac{1}{8} - \frac{2}{64}\right)\right]$$

$$= \left[1 - \frac{1}{32}\right] + \left[-\ln 2 - \frac{1}{2} + \ln 8 + \frac{1}{32}\right]$$

$$8 = 2^3$$

$$= 1 - \ln 2 - \frac{1}{2} + 3\ln 2$$

$$= \frac{1}{2} + 2\ln 2$$