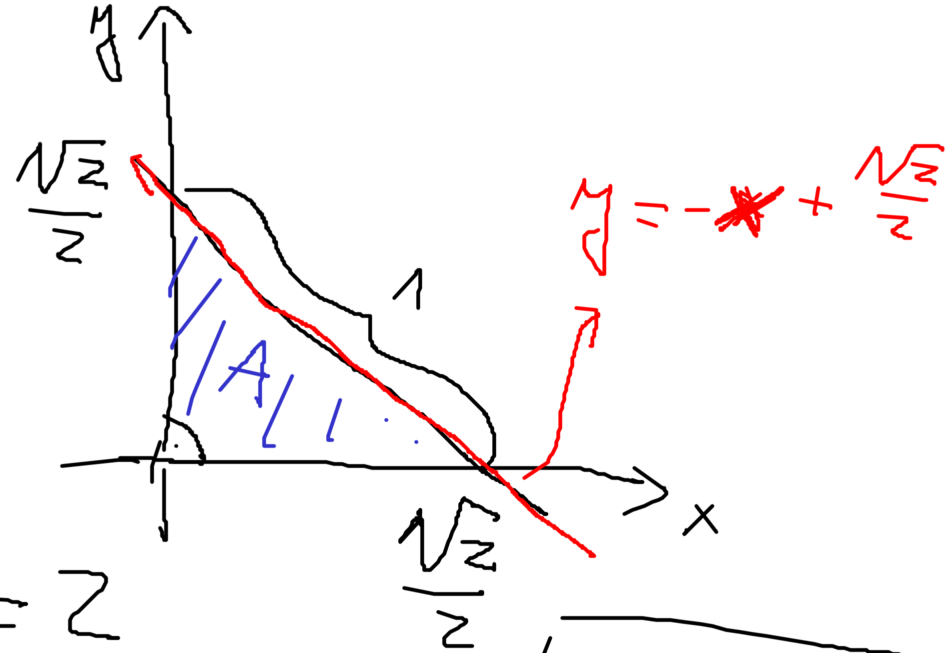
5.7



$$\rho(x,y) = k\,y$$

$$\rho\left(x, \frac{\sqrt{2}}{2}\right) = k\frac{\sqrt{2}}{2} = 2$$

$$k = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\boxed{\rho(x,y) = 2\sqrt{2}\,y}$$

$$M = \iint_A \rho(x,y)\,dx\,dy =$$

$$= \int_{x=0}^{\frac{\sqrt{2}}{2}} \int_{y=0}^{y=-x+\frac{\sqrt{2}}{2}} 2\sqrt{2}\,y\,dx.$$

$$= 2\sqrt{2} \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{2}\left[y^2\right]_0^{-x+\frac{\sqrt{2}}{2}} dx =$$

$$= \sqrt{2} \int_0^{\sqrt{2}/2} \left(x - \frac{\sqrt{2}}{2}\right)^2 dx =$$

$$= \sqrt{2}\left[\frac{1}{3}\left(x-\frac{\sqrt{2}}{2}\right)^3\right]_0^{\frac{\sqrt{2}}{2}} =$$

$$= \sqrt{2}\left[-\frac{1}{3}\left(-\frac{\sqrt{2}}{2}\right)^3\right] =$$

$$= \frac{1}{3}\sqrt{2}\,\frac{2\sqrt{2}}{8} = \frac{1}{6} = M$$ <u>hmotnost</u>

Souřadnice těžiště $[x_0, y_0]$:

$$x_0 = \frac{1}{M}\iint_A x\,\rho(x,y)\,dx\,dy =$$

$$= \frac{1}{M}\int_0^{\frac{\sqrt{2}}{2}}\left(\int_0^{-x+\frac{\sqrt{2}}{2}} 2\sqrt{2}\,x y\,dy\right) dx =$$

$$= \frac{1}{M}\int_0^{\sqrt{2}/2}\left[\sqrt{2}\,x\cdot y^2\right]_{y=0}^{y=-x+\frac{\sqrt{2}}{2}} dx$$

$$= 6\sqrt{2}\int_0^{\sqrt{2}/2} x\left(x-\frac{\sqrt{2}}{2}\right)^2 dx =$$

$$= 6\sqrt{2}\int_0^{\sqrt{2}/2} x\left(x^2 - \sqrt{2}x + \frac{1}{2}\right)dx$$

$$= 6\sqrt{2}\int_0^{\sqrt{2}/2}\left(x^3 - \sqrt{2}\,x^2 + \frac{1}{2}x\right)dx$$

$$= 6\sqrt{2}\left[\frac{1}{4}x^4 - \frac{\sqrt{2}}{3}x^3 + \frac{1}{4}x^2\right]_0^{\frac{\sqrt{2}}{2}}$$

$$= 6\sqrt{2}\left[\frac{1}{4}\cdot\frac{1}{4} - \frac{\sqrt{2}}{3}\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{4}\cdot\frac{1}{2}\right]$$

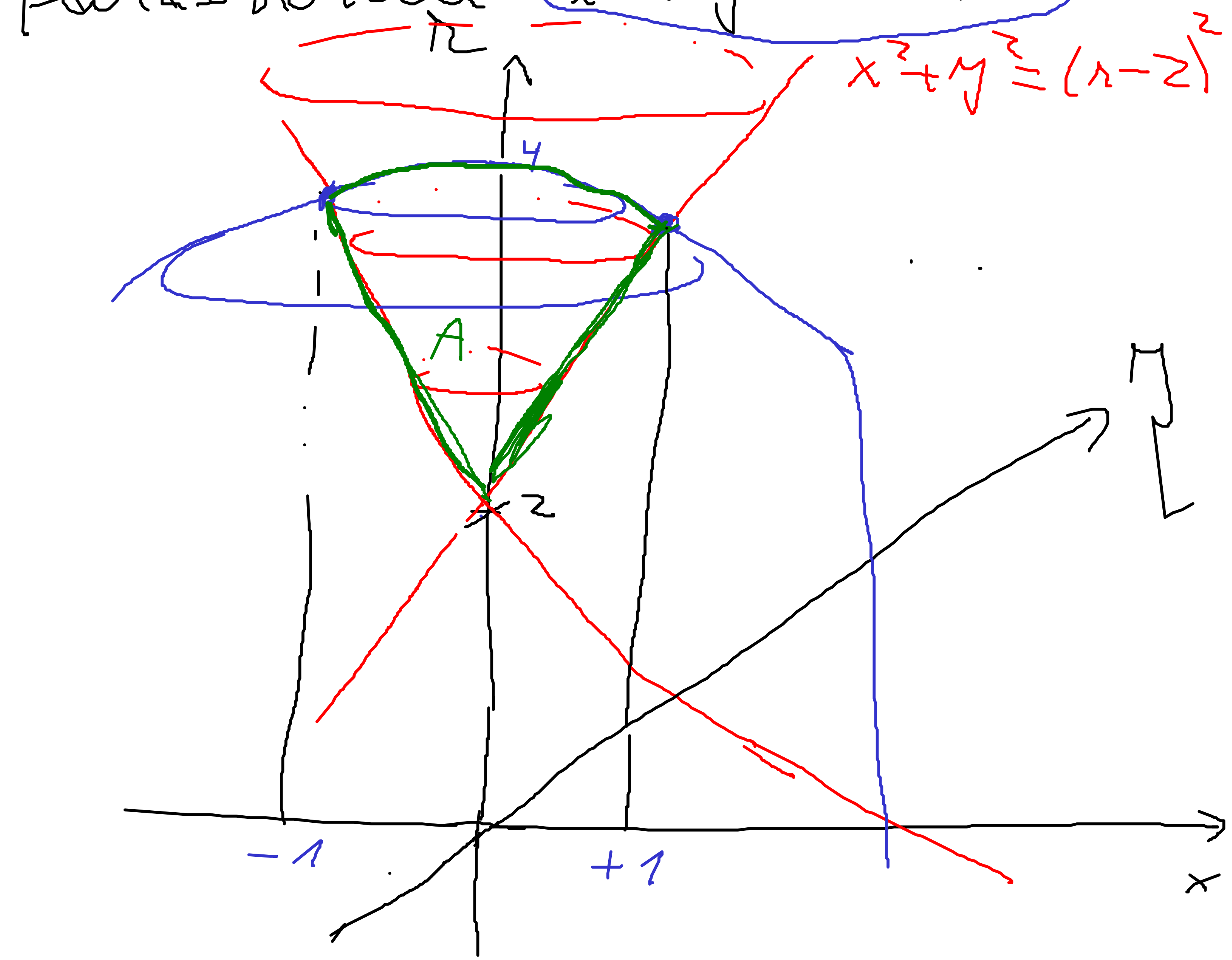
$$= 6\sqrt{2}\left(\frac{1}{16} - \frac{1}{6} + \frac{1}{8}\right) =$$

$$= \sqrt{2}\left(\frac{3}{8} - 1 + \frac{3}{4}\right) = \sqrt{2}\,\frac{3 - 8 + 6}{8}$$

$$= \frac{\sqrt{2}}{8} = x_0$$

$$y_0 = \frac{\sqrt{2}}{4} \qquad T = \left[\frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{4}\right]$$

6.1 Těleso v $\mathbb{R}^3$ ohraničené kuželem $x^2+y^2=(z-2)^2$ $\rightarrow r^2=(z-2)^2$ a paraboloidem $x^2+y^2=4-z$ $\rightarrow r^2=4-z$



$y=0 \rightsquigarrow z=-x^2+4$

$z=-(x^2+y^2)+4$ obecně

$y=0,\ z=-x^2+4$ & $x^2=(z-2)^2$

$x=\pm(z-2)$

$$z = -(z-2)^2 + 4$$

$$z = -(z^2 - 4z + 4) + 4$$

$$z = -z^2 + 4z$$

$$z^2 - 3z = 0$$

$$z(z-3) = 0 \quad \begin{array}{l} \nearrow z = 0 \\ \searrow z = 3 \end{array}$$

$$z = -x^2 + 4$$

$$3 = -x^2 + 4 \qquad x = \pm 1$$

$$M = \iiint_A dx\,dy\,dz$$

Transformace souřadnic

$$(x, y, z) = G(r, \varphi, w) \qquad z = w$$

$$x = r\cos\varphi \qquad \det G' = r$$

$$y = r\sin\varphi$$

$$z = w$$

$$G' = \left(\begin{array}{cc|c} //// & & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$0 \le r \le 1$$
$$0 \le \varphi \le 2\pi$$

$$r+2 \le w \le -r^2+4$$

$$M = \int_{\varphi=0}^{2\pi} \int_{r=0}^{1} \left(\int_{w=r+2}^{w=-r^2+4} r \, dw \right) dr \, d\varphi =$$

$$= \left(\int_{\varphi=0}^{2\pi} d\varphi \right) \left(\int_{r=0}^{1} r \left[w \right]_{r+2}^{-r^2+4} dr \right)$$

$$= 2\pi \int_0^1 r \left(-r^2+4 - (r+2) \right) dr$$

$$= 2\pi \int_0^1 r \left(-r^2 - r + 2 \right) dr$$

$$= 2\pi \int_0^1 \left(-r^3 - r^2 + 2r \right) dr$$

$$= 2\pi \left[-\frac{1}{4} r^4 - \frac{1}{3} r^3 + r^2 \right]_0^1$$

$$= 2\pi \left(-\frac{1}{4} - \frac{1}{3} + 1 \right) = 2\pi \, \frac{-3-4+12}{12}$$

$$= \frac{5\pi}{6}$$

6.2 $y = y(x)$ t. ž.

$$(1+e^x)\, y y' = e^x$$

- Najděte obecné řešení
- — " — řešení splňující $y(0)=1$

$$(1+e^x)\, y \frac{dy}{dx} = e^x$$

$$y\,dy = \frac{e^x}{1+e^x}\,dx \quad / \int$$

$\hookrightarrow$ separovaný tvar

$$\frac{1}{2} y^2 = \int \frac{e^x}{1+e^x}\,dx = \left| \begin{matrix} w = e^x \\ dw = e^x dx \end{matrix} \right|$$

$$= \int \frac{1}{1+w}\,dw =$$

$$= \ln|w+1| + C_1 =$$

$$= \ln(e^x+1) + C_1$$

Obecné řešení: $\boxed{\frac{1}{2} y^2 = \ln(e^x+1) + C}$

$$\cdot \ln(\underbrace{e^x+1}_{\geq 1}) \geq 0 , \quad C_1 \in \mathbb{R}$$

$$\frac{1}{2}y^2 = \ln(e^x+1) + \underbrace{\ln C_2}_{C_1}$$

$C_2 = e^{C_1}$

$C_2 > 0$

$$\frac{1}{2}y^2 = \ln\left(C_2(e^x+1)\right)$$

$$\cdot \; C_2 \geq 1 \Rightarrow C_2(e^x+1) \geq 1$$

$$\Rightarrow \ln\left(C_2(e^x+1)\right) \geq 0$$

$$\Rightarrow D(y) = \mathbb{R}$$

$$y = \pm\sqrt{2\ln\left(C_2(e^x+1)\right)}$$

$$\cdot \; C_2 \in (0,1)$$

$$C_2(e^x+1) \geq 1$$

$$e^x \geq \frac{1}{C_2} - 1 \quad / \ln$$

$$x \geq \ln\left(\frac{1}{C_2} - 1\right)$$

$$D(y) = \left[\ln\left(\frac{1}{c_2} - 1\right), \infty\right)$$

- Počáteční podm. $y(0) = 1$

$$\frac{1}{2}y^2 = \ln(c_2(e^x+1)) \quad \Leftarrow x = 0$$

$$\frac{1}{2} = \ln(2c_2) \quad / e^{()}$$

$$\sqrt{e} = 2c_2 \Rightarrow c_2 = \frac{\sqrt{e}}{2} < 1$$

Výsledek: $\frac{1}{2}y^2 = \ln\left(\frac{\sqrt{e}}{2}(e^x+1)\right)$

$$y = \pm\sqrt{2\ln\left(\frac{\sqrt{e}}{2}(e^x+1)\right)}$$

$$D(y) = \left[\ln\left(\frac{2}{\sqrt{e}} - 1\right), \infty\right)$$

6.3 Řešte rovnici

$$y' = x - \frac{2y}{x^2-1}$$

$y(0) = -1$

$y(2) = 3$

$$y' + \frac{2}{x^2-1}y = x$$

lineární ODR 1 řádu

(1) zhomogenizovaná rovnice

$$y' + \frac{2}{x^2-1}y = 0$$

$$\frac{dy}{dx} = -\frac{2y}{x^2-1}$$

$$-\frac{1}{x^2-1} = \frac{1}{2(x+1)} - \frac{1}{2(x-1)}$$

$\cdot \frac{1}{2y}$ → „ztratili"

$$\frac{dy}{2y} = -\frac{dx}{x^2-1} \quad / \int$$

jsme řešení $y=0$

$$\frac{1}{2}\ln|y| = \frac{1}{2}\int\left(\frac{1}{x+1} - \frac{1}{x-1}\right)dx$$

$$= \frac{1}{2}\left(\ln|x+1| - \ln|x-1|\right) + c$$

$$= \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C_1$$

$$\ln|y| = \ln\left|\frac{x+1}{x-1}\right| + 2C_1 \quad / e^{(\,)}$$

$$|y| = \underbrace{e^{2C_1}}_{C_2>0} \cdot \left|\frac{x+1}{x-1}\right|$$

$$y = C_2 \frac{x+1}{x-1} \qquad C_2 \neq 0$$

lépe $C_2 \in \mathbb{R}$

zbytek příště