1a) $\lim\limits_{\substack{x\to-\infty\\ y\to\infty}} f(x,y) = -\infty$ $\quad$ $\forall \varepsilon>0\ \exists \delta>0\ \forall [x,y]\in D_f:$

$x < -\frac{1}{\delta} \wedge y > \frac{1}{\delta} \Rightarrow f(x,y) < -\frac{1}{\varepsilon}$

b) $\lim\limits_{\substack{x\to-3\\ y\to-\infty}} f(x,y) = 7$ $\quad$ $\forall \varepsilon>0\ \exists \delta>0\ \forall [x,y]\in D_f:$

$|x+3|<\delta \wedge y < -\frac{1}{\delta} \Rightarrow |f(x,y)-7| < \varepsilon$

$$f'_y(x_0,y_0) = \lim_{h\to0} \frac{f(x_0,y_0+h) - f(x_0,y_0)}{h}$$

2) Dle definice $f'_y(3,2) = x^2y + e^x$

$$f'_y(3,2) = \lim_{h\to0} \frac{3^2(2+h)+e^2-(3^2\cdot 2+e^2)}{h} = \lim_{h\to0} \frac{9h}{h} = 9$$

3) $f''_{yx}(4,-5)$ $\quad$ $f''_{yx}(x_0,y_0) = \lim\limits_{h\to0} \dfrac{f'_y(x_0+h,y_0) - f'_y(x_0,y_0)}{h}$

$$\boxed{f''_{yx}(4,5) = \lim_{h\to0} \frac{f'_y(4+h,-5) - f'_y(4,-5)}{h}}$$

4) Má fce $f(x,y) = x^2-3xy+2y^2$ v některém bodě diferenciál

$-8\,dx + 4\,dy$? $\quad$ $2x-3y=-8$ $\quad$ $x=20,\ y=16$

$-3x+4y=4$ $\quad$ $\boxed{[20,16]}$

$f'_x = 2x-3y$

$f'_y = -3x+4y$

5) Tečná rovina k $z = \dfrac{x+2}{y+1}$ v bodě $[2,3,?]$ $\quad$ $z(2,3) = \frac{4}{4} = 1$ $\quad$ $[2,3,1]$

$z'_x = \dfrac{1}{y+1}$ $\quad$ $z'_x(2,3) = \frac{1}{4}$ $\quad$ $\tau:\ z-1 = \frac{1}{4}(x-2) - \frac{1}{4}(y-3)$

$z'_y = -\dfrac{x+2}{(y+1)^2}$ $\quad$ $z'_y(2,3) = -\frac{1}{4}$ $\quad$ $\boxed{x-y-4z+5=0}$

6) Určete d^2f pro $f(x,y) = x^3 - x^2y - y^3$ v bodě $[x_0,y_0]$, dx, dy

$f'_x = 3x^2-2xy$ $\quad$ $f''_{xx} = 6x-2y$ $\quad$ $f''_{xy} = -2x$

$f'_y = -x^2-3y^2$ $\quad$ $f''_{yx} = -2x$ $\quad$ $f''_{yy} = -6y$

$$d^2f = f''_{xx} h^2 + 2 f''_{xy}\, h\cdot k + f''_{yy} k^2$$

$$\boxed{d^2f(x_0,y_0) = (6x_0-2y_0)(dx)^2 - 4x_0\, dx\,dy - 6y_0 (dy)^2}$$

7) Řešte $y' - 4xy = 0$ $\quad y=0$ je řešení, $\quad \ln|y| = 2x^2 + c$

$$\frac{dy}{dx} = 4xy \quad /:y \neq 0$$

$$|y| = c\,e^{2x^2} \quad c>0$$

$$\int \frac{dy}{y} = \int 4x\,dx$$

$$\boxed{y = c\,e^{2x^2}, \quad c \in \mathbb{R}}$$

8) Navrhněte substituci pro rovnice, dosaďte, neřešte

a) $(x^2+y^2)\,y' = 4xy$

$$y' = \frac{4xy}{x^2+y^2}$$

$$y' = \frac{4\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}$$

$$u = \frac{y}{x} \to u \cdot x = y \to y' = u'x + u$$

$$u'x + u = \frac{4u}{1+u^2} \longrightarrow \boxed{u'x = \frac{4u - 4u^3}{1+u^2}}$$

separovatelná

b) $y' = \dfrac{6 + x^2y^4}{y^3}$

$$y' = \frac{6}{y^3} + x^2 y$$

$$y' = x^2 y + \frac{6}{y^3} \quad /\cdot y^3$$

$$y^3 y' = x y^4 + 6 \qquad u = y^4 \to u' = 4y^3 y'$$

$$\boxed{\frac{1}{4}u' = xu + 6}$$

lineární dif. rovnice 1. řádu

9) Sestavte rovnici 2. řádu s obecným řešením $y = C_1 e^{2x} + C_2 e^{-x}$

$\lambda_1 = 2, \ \lambda_2 = -1 \qquad (\lambda-2)(\lambda+1) = \lambda^2 - \lambda - 2 = 0$

$$\boxed{y'' - y' - 2y = 0}$$

10) Navrhněte tvar part. řešení rovnice

$$y'' + 4y = x^2 \sin 2x \qquad x^2 \sin 2x \to \mu = 0 \text{ a } \nu = 2$$

číslo $2i$ je kořen

$\lambda^2 + 4 = 0$

$\lambda_{1,2} = \pm 2i = 0 \pm 2i$

$$\boxed{y_0 = x\left[(ax^2+bx+c)\cos 2x + (dx^2+ex+f)\sin 2x\right]}$$

1) $f(x,y) = 3x^2 - 6xy + y^3 - 9y + \pi$, určete extrémy a hodnoty v nich

$f'_x = 6x - 6y$

$f'_y = -6x + 3y^2 - 9$

$x - y = 0 \quad x = y$

$2x - y^2 + 3 = 0$

$2x - x^2 + 3 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x_1 = 3, y_1 = 3$

$x_2 = -1, y_2 = -1$

St. body $[3,3]$, $[-1,-1]$

$f''_{xx} = 6$, $f''_{xy} = -6 = f''_{yx}$, $f''_{yy} = 6y$

$f''_{xx} \cdot f''_{yy} - (f''_{xy})^2$

Pro $[3,3]$ $\quad \underline{f(3,3) = \pi - 27}$

$f''_{xx}(3,3) = 6 \quad f''_{xy}(3,3) = -6$

$f''_{yy}(3,3) = 18$

$6 \cdot 18 - (-6)^2 > 0 \rightarrow$ je extrém

$f''_{xx}(3,3) = 6 > 0 \rightarrow$ $\underline{[3,3] \text{ je LMax}}$

Pro $[-1,-1]$

$f''_{xx}(-1,-1) = 6$, $f''_{xy}(-1,-1) = -6$, $f''_{yy}(-1,-1) = -6$

$6 \cdot (-6) - (-6)^2 < 0$ $\quad$ <u>není extrém</u>

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2) Řešte $y' \sqrt{1+x^2} = \frac{x}{y}$, $y(0) = -2$

$\frac{dy}{dx} \sqrt{1+x^2} = \frac{x}{y}$

$\int y\,dy = \int \frac{x}{\sqrt{1+x^2}}\,dx$

$\int \frac{x}{\sqrt{1+x^2}}\,dx = \left| \begin{array}{l} 1+x^2 = t \\ 2x\,dx = dt \end{array} \right| = \frac{1}{2} \int \frac{dt}{\sqrt{t}} =$

$= \frac{1}{2} \frac{\sqrt{t}}{\frac{1}{2}} = \sqrt{t} = \sqrt{1+x^2} + C$

$\frac{1}{2} y^2 = \sqrt{1+x^2} + C$

$\boxed{y^2 = 2\sqrt{1+x^2} + C}$

$y = \pm\sqrt{2\sqrt{1+x^2} + C}$

$y(0) = -2$, kdy jen

$y = -\sqrt{2\sqrt{1+x^2} + C}$

$-2 = -\sqrt{2 + C} \rightarrow \underline{\underline{C = 2}}$

$$\boxed{y = -\sqrt{2\left(1 + \sqrt{1+x^2}\right)}}$$

3) $4y'' + 12y' + 9y = 9x^2 + 24x + 8$

$$4y'' + 12y' + 9y = 0$$

$$4\lambda^2 + 12\lambda + 9 = 0$$

$$(2\lambda + 3)^2 = 0$$

$$\lambda = -\frac{3}{2}$$

$$\boxed{y_H = C_1 e^{-\frac{3}{2}x} + C_2 x e^{-\frac{3}{2}x}}$$

$$y_0 = ax^2 + bx + c$$

$$y_0' = 2ax + b$$

$$y_0'' = 2a$$

$$8a + 24ax + 12b + 9ax^2 + 9bx + 9c = 9x^2 + 24x + 8$$

$$9ax^2 + (24a + 9b)x + 8a + 12b + 9c = 9x^2 + 24x + 8$$

$$9a = 9 \longrightarrow \underline{\underline{a = 1}}$$

$$24 + 9b = 24 \longrightarrow \underline{\underline{b = 0}}$$

$$8 + 9c = 8 \longrightarrow c = 0$$

$$\boxed{y_0 = x^2}$$

$$\boxed{y_{WH} = C_1 e^{-\frac{3}{2}x} + C_2 x e^{-\frac{3}{2}x} + x^2}$$